ON THE GROUP OF WEAK AUTOMORPHISMS OF A FAMILY OF EQUIVALENCE RELATIONS

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Abstract Let E be a set of equivalence relations on the set Ω. We call a permutation π of Ω an *e-permutation*, $e \in E$, if it leaves every e-class of Ω invariant. We consider the set $S(E) := \bigcup_{e \in E} S(e)$ of all *E-permutations* of Ω or *weak automorphisms* of E. If E is directed by the natural partial order the set $S(E)$ is a group - the *E-symmetric group* - our *group of weak automorphisms* of E. If the family E is well-behaved (*homogeneous*) and Ω is countable then the class of all E-symmetric groups can be characterised up to permutation-isomorphism by the *Steinitz type*. Some normal subgroups of $S(E)$ are pointed out.

1 Introduction

Let Ω be an infinite set and e an equivalence relation on Ω. The permutation π of Ω is an *e-permutation* if it leaves every e-class of Ω invariant. The set $S(e)$ of all e-permutations is a subgroup of $S(\Omega)$, the symmetric group on Ω; it is isomorphic to the Cartesian product of the symmetric groups on the e-classes of Ω. If E is a set of equivalence relations on Ω, we shall consider the set $S(E) := \bigcup_{e \in E} S(e)$ of all *E-permutations* of Ω or *weak E-automorphisms*.

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If we endow the family E with some smoothness properties (*homogeneity*) and if Ω is countable such sets of equivalence relations are classified in a *three-class-society*. Further, the class of all E-symmetric groups will be characterised up to permutation-isomorphism by the *Steinitz type*. Similar applications of *Steinitz arguments* are presented by Kroshko/Sushchansky in [5] for the complete classification of limit groups of countable direct systems of finite symmetric groups with strictly diagonal embeddings.

As subgroups of the symmetric group $S(\Omega)$ on Ω the groups $S(E)$ contain $S_0(\Omega)$ and $Alt(\Omega)$, the only non-trivial normal subgroups of $S(\Omega)$ if Ω is countable. In this case we point out a few further normal subgroups and their relative position in $S(E)$. Unfortunately, we are far from describing all the normal subgroups of $S(E)$.

2 Foundations

Consider the set $S(E)$ of all *E-permutations* on an infinite set Ω as mentioned in the section above. On the set E of equivalence relations on Ω there is a natural partial order. For equivalence relations e and e' we shall say: e' is *coarser* than e ($e' \geq e$) if every e'-class of Ω is the union of e-classes. If for the relations e, e' one has $e' \geq e$ then one has $S(e') \supseteq S(e)$. The following four propositions are immediate:

Proposition 2.1 *If the set E of equivalence relations on Ω is directed, i.e. if to any pair $e', e'' \in E$ there is an $e \in E$ with $e \geq e'$ and $e \geq e''$, then the set $S(E)$ is a subgroup of $S(\Omega)$.*

The set E of equivalence relations on Ω is *transitive* if for any two elements $\omega, \omega' \in \Omega$ there is an $e \in E$ such that ω and ω' are in the same e-class of Ω ($\omega \sim_e \omega'$).

Proposition 2.2 *If the set E of equivalence relations on Ω is directed and transitive then, for every $n \in I\!N$, the group $S(E)$ acts n-transitively on Ω; the group $S_0(\Omega)$ of all finitary permutations of Ω is a normal subgroup of $S(E)$.*

Two sets E and E' of equivalence relations on Ω are *of the same type* if for every $e \in E$ there is an $e' \in E'$ with $e' \geq e$ and if for every $e' \in E'$ there is an $e \in E$ with $e \geq e'$.

Proposition 2.3 *If the sets E and E' of equivalence relations on Ω are of the same type then the sets $S(E)$ and $S(E')$ of permutations of Ω coincide.*

For the set E of equivalence relations on Ω and $e \in E$ define the subset $E_e := \{e' \in E | e' \geq e\}$.

Proposition 2.4 *If the set E of equivalence relations on Ω is directed then, for every $e \in E$, the subset E_e of E is directed and of the same type as E.*

The equivalence relation e on Ω is *finite (bounded)* if every e-class of Ω has only finitely (boundedly) many elements and the set E is *finite (bounded)* if every $e \in E$ is finite (bounded). If there is an infinite e-class of Ω then e as well as E is called *infinite*. If the equivalence relation e' on Ω is coarser than the equivalence relation e, then e' is called a *finite (bounded) coarsening* of e if every e'-class of Ω is the union of finitely (boundedly) many e-classes. If there is an e'-class that is union of infinitely many e-classes, we shall call e' an *infinite coarsening* of e.

If the set E of equivalence relations on Ω is directed one can characterise some properties of E by properties of the group $S(E)$.

Theorem 2.5 *If the set E of equivalence relations on Ω is directed one has*

(1) The group $S(E)$ is torsion if and only if every $e \in E$ is bounded. In this case $S(E)$ is a locally finite group.

(2) The group $S(E)$ is locally residually finite if and only if every $e \in E$ is finite.

(3) There is an infinite $e \in E$ if and only if $S(E)$ contains an infinite, finitely generated subgroup which is not residually finite. In this case the subset E_e, which is of the same type as E, consists of infinite equivalence relations on Ω only.

Proof: (1) If $S(E)$ is a torsion group then so is every $S(e)$, $e \in E$. Assuming that the e-classes are not of bounded size, one obtains cycles of unbounded size and support in distinct e-classes. In the Cartesian product $S(e)$ these cycles yield an element of infinite order. Contradiction. If, on the other hand the equivalence relation $e \in E$ is bounded, $S(e)$ has finite exponent. As Cartesian product of finite groups of bounded order $S(e)$ is locally finite, see [6].

(2) Let G be a finitely generated subgroup of $S(E)$, and assume that every $e \in E$ is finite. Then there is an $e \in E$ with $G \subseteq S(e)$. Thus, as a subgroup of the Cartesian product $S(e)$ of finite symmetric groups, G is residually finite.

(3) Let $e \in E$ have an infinite class $[\omega]_e$ in Ω. Since there are infinite, finitely generated groups which are not residually finite, see [2], one can embed them (by the right regular representation) into the symmetric group $S([\omega]_e)$, hence into $S(e) \subseteq S(E)$. With e every $e' \geq e$ is infinite.

Considering sets E of equivalence relations on Ω, it is useful to observe that they come in two types:

Proposition 2.6 *A given set E of equivalence relations on Ω is either an infinite coarsening if for every $e \in E$ there is an infinite coarsening $e' \in E$ of e, or E is a finite coarsening if there is an element $e \in E$ such that every coarsening $e' \in E$ of e is a finite coarsening of e.*

This dichotomy is particularly useful if E is directed; we are going to refine it in the next section.

Up to the end of this section we suppose that E has a strictly ascending sequence $T = (t_i)_{i \in \mathbb{N}}$ of the same type as E; in this case T is said to be an *approximating tower* of E. Thus we may restrict attention to a suitable approximating tower.

If two e-classes on Ω have the same cardinality, we say that they have the same *class-type*. Recall that if for $e' \geq e$ every e'-class is union of e-classes of in general different class-types. If we group together every t_i-class contained in $[\omega]_{t_{i+1}}$, seperately for every (t_i-) class-type appearing, the different cardinalities of the resulting unions form the *level-type* of $[\omega]_{t_{i+1}}$; another description of $|[\omega]_{t_{i+1}}|$ which refers to the partition of $[\omega]_{t_{i+1}}$ in its t_i-classes of different class-types. If we have for E, respectively E', two approximating towers $T = (t_i)_{i \in \mathbb{N}}$, respectively $T' = (t'_i)_{i \in \mathbb{N}}$, we are interested in the question whether a t_{i+1}-class and a t'_{i+1}-class have the *same level-type*, i.e. the distinct unions of t_i- respectively t'_i-classes of each class-type appearing, have the same cardinality in each case. We call $T = (t_i)_{i \in \mathbb{N}}$ and $T' = (t'_i)_{i \in \mathbb{N}}$ *type-isomorphic* if for every $i \geq 1$ there are equivalence classes of t_i and t'_i of the same level-type. From the following theorem we obtain necessary conditions for two sets of equivalence relations to be *permutation-equivalent*, i.e. they are transformed into one another by a permutation of $S(\Omega)$.

Theorem 2.7 *Let E and E' be sets of equivalence relations with approximating towers $T = (t_i)_{i \in \mathbb{N}}$ and $T' = (t'_i)_{i \in \mathbb{N}}$, defined on two underlying sets Ω and Ω', respectively. If T and T' are type-isomorphic then:*

$$(S(E), \Omega) \simeq (S(E'), \Omega').$$

Proof: First construct a bijection β, step by step, of the underlying sets with respect to the different partitions in every step. Since T and T' are type-isomorphic there exist a t_1-class and a t'_1-class having the same level-type. Choose arbitrary elements $\omega \in \Omega$, $\omega' \in \Omega'$ of a t_0- respectively t'_0-class of the same class-type and fix them. Define the bijection β_0 by $[\omega]_{t_0} \to [\omega']_{t'_0}$. Assuming the bijection β_i has been already chosen, adjust β_{i+1} to an extension of the bijection β_i on $[\omega]_{t_{i+1}}$ in the following way: β_{i+1} maps every t_i-class contained in $[\omega]_{t_{i+1}}$ onto a t'_i-class of the same class-type in $[\omega']_{t'_{i+1}}$. By assumption $[\omega]_{t_{i+1}}$ and $[\omega']_{t'_{i+1}}$ have the same level-type, so β_{i+1} is an one-to-one correspondence between t_i-classes and t'_i-classes of the same class-type. Hence β_{i+1} is a bijection which preserves every t_j-class (and every t'_j-class) for $j \leq i$ in $[\omega]_{t_{i+1}}$, respectively $[\omega']_{t'_{i+1}}$. Iterating this procedure, one gets β as a limit of expanding bijections β_i and therefore β is a bijection.

In the second part of the proof we have to construct a suitable group-isomorphism $\alpha : S(E) \longrightarrow S(E')$ so that the following property holds for all $\omega \in \Omega$ and

$\pi \in S(E)$: $\omega^{\pi\beta} = \omega^{\beta\pi^{\alpha}}$. For every $\pi \in S(E)$ there exists an $t_i \in T \subseteq E$ so that $\pi \in \prod_{\omega\in\Omega} S([\omega]_{t_i})$. By the action of β_i from the first part π induces a permutation $\pi^{\alpha_i} \in \prod_{\omega'\in\Omega'} S([\omega']_{t'_i})$ because any two elements of the same t'_i-class have their images in a t_i-class under the permutation $\pi^{\alpha_i} = \beta_i^{-1}\pi\beta_i$. Hence we get:

$$\prod_{\omega\in\Omega} S([\omega]_{t_i}) \overset{\alpha_i}{\cong} \prod_{\omega'\in\Omega'} S([\omega']_{t'_i}).$$

Finally, if we define α as the limit of the expanding group-isomorphisms α_i, $i \in I\!N$, this yields the isomorphism since

$$S(E) = \bigcup_{i\in I\!N}\prod_{\omega\in\Omega} S([\omega]_{t_i}) \overset{\alpha}{\cong} \bigcup_{i\in I\!N}\prod_{\omega'\in\Omega'} S([\omega']_{t'_i}) = S(E')$$

and because E, T respectively E', T' are of the same type.

3 Characterisations

Theorem 2.7 suggests a programme for describing isomorphism types of all E-symmetric groups, i.e. to distinguish different approximating towers. For answering the underlying question what kind of influence has a set E of equivalence relations defined on Ω on the structure of the affiliated group of E-permutations the latest theorem instructs us to classify the sets E respectively the approximating towers T which are of the same kind. If we consider from now on sets of equivalence relations defined on Ω having some smoothness properties we get a remarkable *three-class-society* describing such sets of equivalence relations.

First it is useful to describe the underlying set Ω via ascending families of e-classes. If the set E is directed we write $\Omega = \bigcup_{e\in E}[\omega]_e$, for all $\omega \in \Omega$; call such an ascending family $\{[\omega]_e\}_{e\in E}$ an *approximating family* of Ω. From now on we suppose that E has an approximating tower T. Hence Ω is covered by an *approximating sequence* $\{[\omega]_{t_i}\}_{i\in I\!N}$, for $t_i \in T$. the equivalence relation e is said to be *balanced* if all e-classes have the same cardinality, and further we call E ($\aleph_0$-)*balanced*, if every $e \in E$ is balanced (and all e-classes are countable at most).

Now let E be $\aleph_0$-balanced and have an approximating tower, then Ω is countable. For $e' \geq e$ we define the *index* $|e' : e|$ *of* e *in* e': In case every e'-class of Ω is the union of exactly n e-classes of Ω for some $n \in I\!N$ then $|e' : e| = n$, otherwise $|e' : e| = \aleph_0$. For short, we call the $\aleph_0$-balanced set E of equivalence relations having an approximating tower *homogeneous* if the index can be defined for any two $e, e' \in E$ with $e' \geq e$.

If E is homogeneous every approximating family of the underlying set Ω belongs to one of the following classes of a *three-class-society*:

1. For every $e \in E$ there exists a coarser relation $e' \in E$ with $|e' : e| = \aleph_0$.
2. In E there exists a relation e so that for every relation $e' \in E$ coarser than e one has: $|e' : e| < \aleph_0$ and $|[\omega]_e| = \aleph_0$.
3. For any two relations e and e' of E with $e' \geq e$ one has: $|e' : e| < \aleph_0$ and $|[\omega]_e|$ is finite.

Now we may translate the necessary condition of theorem 2.7 to the case of homogeneous sets of equivalence relations. For every $e \in E$ there is only one class-type hence the approximating towers T and T' are type-isomorphic if and only if there are equipotent equivalence classes $[\omega]_{t_0}$ and $[\omega']_{t'_0}$, respectively, and every t_i and t'_i has the same index in any coarser equivalence relation t_j, respectively t'_j, for $t_i, t_j \in T$ and $t'_i, t'_j \in T'$.

Corollary 3.1 *Let E and E' be sets of homogeneous equivalence relations with approximating towers $T = (t_i)_{i \in \mathbb{N}}$ and $T' = (t'_i)_{i \in \mathbb{N}}$, defined on two sets Ω, respectively Ω'; for both underlying sets there are approximating sequences $\{[\omega]_{t_i}\}_{i \in \mathbb{N}}$ and $\{[\omega']_{t'_i}\}_{i \in \mathbb{N}}$, for $t \in T$, $t' \in T'$ and $\omega \in \Omega$, $\omega' \in \Omega'$ with at least two equipotent equivalence classes $[\omega]_{t_0}$, respectively $[\omega']_{t'_0}$. If these towers have the same index sequences $\{|t_{i+1} : t_i|\}_{i \in \mathbb{N}}$, respectively $\{|t'_{i+1} : t'_i|\}_{i \in \mathbb{N}}$, then*

$$(S(E), \Omega) \simeq (S(E'), \Omega').$$

Hence we get immediately a complete description for the Upper Class of the above society, since for every $t_i \in T$ there exists a coarser relation $t_j \in T$ with $|t_j : t_i| = \aleph_0$.

Corollary 3.2 *For any two sets E, E' of the Upper Class the two groups $S(E)$ and $S(E')$ with their actions on Ω and Ω', respectively, are permutation-equivalent.*

What can be said about the group $S(E)$ of all E-permutations if E does not belong to the Upper Class? To examine this question further we want to construct a group in which a given approximating tower T can be realised.

It will be convenient to consider as the underlying set Ω a countable abelian group A which is not finitely generated. Clearly A is union of its finitely generated subgroups A_e, for $e \in \mathbb{N}$. For every index e we define an equivalence relation in the following way: $x \sim_e y$ holds if and only if $x = ya$ for some $a \in A_e$. Note, that in this case E is homogeneous and has an approximating tower $T = (t_i)_{i \in \mathbb{N}}$ such that $A = \bigcup_{i \in \mathbb{N}} A_{t_i}$. For a fixed approximating tower T we construct a corresponding abelian group $A(T) =: A$ as follows:

Denote for short by $A_i := A_{t_i}$ and let $A_1 := \mathbb{Z}_{|t_1 : t_0|}$, the cyclic group of order $|t_1 : t_0|$. Let $A_0 = 1$, the identity subgroup; if A_i is already chosen, then inductively define A_{i+1} as $A_{i+1} := A_i \oplus \mathbb{Z}_{|t_{i+1} : t_i|}$. But, put $\mathbb{Z}_{|t_{i+1} : t_i|} := \mathbb{Z}$, if and only if $|t_{i+1} : t_i| = |A_{i+1} : A_i|$ is infinite. Put $A := \bigcup_{i \in \mathbb{N}} A_i$. This yields a *approximating sequence of subgroups* $\{A_i\}_{i \in \mathbb{N}}$ of A which meet all requirements, because A is a direct sum of finite cyclic groups, possibly together with infinite cyclic groups.

With such a realisation an arbitrary e-permutation π of A leaves every left coset of A_e invariant and hence every left coset of A_i for almost all $i \in \mathbb{N}$. Therefore one gets on the underlying set A a subgroup of the symmetric group, the *constricted symmetric group on A*. Actually this group has been introduced by B.A.F. Wehrfritz in [4, chapter VI]. Recall the slightly more general definition (cf. [3, S.297]) of the constricted symmetric group on an arbitrary group G:

$CS(G) = \{\pi \in S(G)|$ for some finitely generated subgroup G_π of G one has $(xG_\pi)^\pi = xG_\pi$ for all $x \in G\}$.

Notice that $CS(G)$ is isomorphic to the union of Cartesian products of the symmetric groups on left cosets with respect to all finitely generated subgroups of G. - Summing up, one gets

Theorem 3.3 *If E is homogeneous with an approximating tower $T = (t_i)_{i \in \mathbb{N}}$ then there is a countable, infinitely generated abelian group A such that:*

$$(S(E), \Omega) \simeq (CS(A), A).$$

Observe that $CS(A)$ does not depend on the group structure of A nor on the particular approximating tower. Even if the steps in the construction of the approximating tower $T = (t_i)_{i \in \mathbb{N}}$ will be permuted by σ, the resulting tower $T_\sigma = (t_{\sigma(i)})_{\sigma(i) \in \mathbb{N}}$ still yields the same group since A is abelian.

What is the incisive information distinguishing non-isomorphic constricted symmetric groups? One possibility is to search for invariants of different approximating towers, respectively approximating group sequences, yielding the same constricted symmetric group. Since A is abelian every tower can be refined if necessary by inserting for each step finitely many. Therefore assume that the approximating sequences have only steps of *step size* $|A_{i+1} : A_i|$ a power of p or infinite ones.

Moreover, a countable, infinitely abelian group $A = \bigcup_{i \in \mathbb{N}} A_i$, respectively its approximating sequence $\{A_i\}_{i \in \mathbb{N}}$ of subgroups *belongs to the same class* of the three-class-society as its corresponding approximating sequence $\{[\omega]_{t_i}\}_{t_i \in T}$ does. The countably generated free abelian group is an example for a group of the Upper Class. Clearly, countable elementary abelian (abelian locally finite) groups are examples for groups of the Lower Class. Note that, in this case, applying theorem 2.5, the affiliated constricted symmetric groups are also locally finite. If L is a non-trivial group in the Lower Class then $L \oplus \mathbb{Z}$ is in the Middle Class.

Decisive invariants for approximating sequences of subgroups are the number of different p-steps for any prime p, appearing in the affiliated index sequence $\{|A_{i+1} : A_i|\}_{i\in\mathbb{N}}$. Therefore one defines as a new measurement the *Steinitz type (supernatural number)* $St(\{A_i\}_{i\in\mathbb{N}})$ *of* A, the sequence of exponents appearing of all p-steps in a chosen approximating sequence of subgroups $\{A_i\}_{i\in\mathbb{N}}$ of A. Denote the Steinitz type of A by $St(\{A_i\}_{i\in\mathbb{N}}) := (n_\infty; n_2, n_3, n_5, \ldots, n_p, \ldots\ldots)$, whereby the coordinate n_p, for a prime p, stands for the existance of a p^{n_p}-step in the index sequence $\{|A_{i+1} : A_i|\}_{i\in\mathbb{N}}$. Further $n_\infty \in \{0,1\}$ and $n_\infty = 1$, when ever there is at least an infinite step in the index sequence $\{|A_{i+1} : A_i|\}_{i\in\mathbb{N}}$ and $n_\infty = 0$ otherwise. We say for two groups A and B that their Steinitz types $St(\{A_i\}_{i\in\mathbb{N}})$ and $St(\{B_i\}_{i\in\mathbb{N}})$ are *equal* if their entries coincide.

As a consequence of theorem 2.7 equal Steinitz types lead to constricted symmetric groups which are permutation-equivalent and thus are isomorphic as abstract groups. Conversely, assume for groups A and B of the Lower Class (recall: as groups belonging to one class of the three-class-society they are countable, infinitely generated abelian) that $St(\{A_i\}_{i\in\mathbb{N}}) \neq St(\{B_i\}_{i\in\mathbb{N}})$. Hence there is at least one coordinate n_p in which their Steinitz types differ. Now, let n be the entry for n_p in $St(\{A_i\}_{i\in\mathbb{N}})$ and let m be the corresponding entry in $St(\{B_i\}_{i\in\mathbb{N}})$ with $n > m$, for $n, m \in \mathbb{N}$. In $CS(A)$ there is a permutation π and a subgroup A_π of A of order p^n such that π acts regularly on every left coset xA_π, for all $x \in A$. There isn't such a permutation in $CS(B)$, since $n > m$. Therefore $(CS(A), A) \not\simeq (CS(B), B)$. This proves the following

Theorem 3.4 *For any two groups A, B in the Lower Class one has:*

$$St(\{A_i\}_{i\in\mathbb{N}}) = St(\{B_i\}_{i\in\mathbb{N}}), \text{ if and only if } (CS(A), A) \simeq (CS(B), B).$$

We have shown that for groups of the Lower Class the Steinitz type characterises the constricted symmetric groups as permutation groups. However the question, whether the Steinitz type of a group of the Lower Class distinguishes the isomorphism types of constricted symmetric groups, i.e. of the $S(E)$, is left open.

For groups of the Middle Class in the three-class-society we are going to show that such a strong characterisation cannot be expected. In this case it is convenient to consider an appropriate approximating sequence of subgroups $\{A_i\}_{i\in\mathbb{N}}$ for a group A starting with an infinite step. Expand every infinite step by combining it with finitely many finite steps to a single infinite step (or shrink every infinite step in a similar way, of course). This manipulation leads to another approximating sequence of subgroups $\{A'_i\}_{i\in\mathbb{N}}$ which has still the same limit A and hence the same constricted symmetric group. But $St(\{A'_i\}_{i\in\mathbb{N}})$ differs now from $St(\{A_i\}_{i\in\mathbb{N}})$ at finitely many entries at most by finite exponent after the semicolon in the notation of the Steinitz type. Thus, Steinitz types of groups belonging to the Middle Class can be modified finitely without yielding another constricted symmetric group. Calling two Steinitz types *equivalent* if they differ finitely at most in finitely many distinct components, we define an equivalence relation on the class of all Steinitz types.

In the light of groups belonging to the Middle Class we ask, whether constricted symmetric groups which are not permutation-equivalent must have non-equivalent Steinitz types?

4 Normal subgroups

Let E be a directed and transitive set of equivalence relations on Ω. Recall from proposition 2.2 that both the group $S(\Omega)$ and $S(E)$, act n-transitively on Ω for every $n \in \mathbb{N}$. Thus, in a certain sense, they act similarly. But how similar are they? One possible approach to this question is to examine the structure of their normal systems. Do they have the same normal subgroups? The normal system of the symmetric group of countable degree is well known, since [7], and since [1] without any restriction on the cardinality of the underlying set Ω. Up to the end of this section we suppose Ω to be countable for focussing not too much on normal subgroups based only on set-theoretic properties. Again from proposition 2.2 we know that the only non-trivial normal subgroups of the symmetric group of countable degree, $S_0(\Omega)$ and $Alt(\Omega)$, the group of all finite even permutations, are also normal subgroups of $S(E)$. Is the factor ${}^{S(E)}/_{S_0(\Omega)}$ simple? The answer is: No, in general.

For $\pi \in S(E)$ and $e \in E$ we call an e-class π*-admissible* if π leaves every e-class of Ω invariant. Further denote by $\pi_{\omega,e}$ the restriction of π on the e-class $[\omega]_e$ and by $supp(\pi)$ the set of all elements in Ω moved by π. We want to call the reader's attention to the following two sets, which occur clearly as normal subgroups of $S(E)$:

Denote by $F(E)$ the set of all E-permutations acting finitely on admissible equivalence classes defined as follows: $F(E) := \{\pi \in S(E) |$ for all π-admissible e-classes one has $|supp(\pi_{\omega,e})| < \aleph_0$, for all $\omega \in \Omega\}$.

Further, let $B(E) := \{\pi \in S(E) |$ for all π-admissible e-classes there exist a $c(\pi, e) \in \mathbb{N}$ with $|supp(\pi_{\omega,e})| < c(\pi, e)$, for all $\omega \in \Omega\}$, the set of all E-permutations whose support on admissible equivalence classes is uniformly bounded by an integer c depending on the permutation π and the equivalence relation $e \in E$.

Note, that $S_0(\Omega)$ is strictly contained in $B(E)$ hence in $F(E)$ since there are permutations with infinite support in $B(E)$. As in theorem 2.5 we conclude that $B(E)$ is locally finite and $F(E)$ is locally residually finite. If E is not bounded there is a permutation in $F(E)$ with a cyclic decomposition in infinite many cycles of different finite degrees. Hence, in this case, we have $B(E) \ntrianglelefteq F(E)$, i.e. $B(E)$ is a *proper normal subgroup* of $F(E)$. If E is bounded we get only $B(E) = F(E)$. In the case that there is an infinite $e \in E$ one gets $F(E) \ntrianglelefteq S(E)$ since there are permutations in $S(E)$ which act regularly on every admissible equivalence class, otherwise $F(E) = S(E)$. Therefore one has the following

Proposition 4.1 *$B(E)$ is a locally finite normal subgroup and $F(E)$ is a locally residually finite normal subgroup of $S(E)$. Their relative positions depend on properties of E:*

1. *If E is bounded then $S_0(\Omega) \ntrianglelefteq B(E) = F(E) = S(E)$.*

2. *If E is finite and unbounded one has: $S_0(\Omega) \ntrianglelefteq B(E) \ntrianglelefteq F(E) = S(E)$.*

3. *If E is infinite one has: $S_0(\Omega) \ntrianglelefteq B(E) \ntrianglelefteq F(E) \ntrianglelefteq S(E)$.*

Further, we consider the set $M(E) := \{\pi \in S(E) |$ there is a $[\omega]_e$ for $\omega \in \Omega$, $e \in E$ such that $supp(\pi) \subseteq [\omega]_e\}$. Since E is transitive and directed $M(E)$ is also a normal subgroup of $S(E)$, and $M(E) = S_0(\Omega)$ if and only if E is finite. Therefore we assume from now on that E is infinite.

Furthermore, if we choose an arbitrary $\omega \in \Omega$ and fix it and assume that there is an equivalence relation $e \in E$ so that $[\omega]_e$ is infinite, $S_0(\Omega) = \bigcup_{e\in E} S_0([\omega]_e)$ and $M(E) = \bigcup_{e\in E} S([\omega]_e)$ holds since E is directed and transitive. From [7] we know that ${}^{S([\omega]_e)}/_{S_0([\omega]_e)}$ is a non-trivial simple factor and moreover for all $e \in E$ we have that $S_0([\omega]_e)$ is a maximal normal subgroup of $S([\omega]_e)$.

Let α be the canonical epimorphism mapping $S([\omega]_{e'})$ onto ${}^{S([\omega]_{e'})}/_{S_0([\omega]_{e'})}$. For every $e \in E$ with $e' \geq e$ α maps $S([\omega]_e)$ onto ${}^{S([\omega]_e)\cdot S_0([\omega]_{e'})}/_{S_0([\omega]_{e'})}$ which is isomorphic to ${}^{S([\omega]_e)}/_{S([\omega]_e)\cap S_0([\omega]_{e'})}$ by the Isomophism Theorem. Since $S_0([\omega]_e)$ is the maximal normal subgroup of $S([\omega]_e)$ we get $S_0 \cap S_0([\omega]_{e'}) = S_0([\omega]_e)$, hence

$${}^{S([\omega]_{e'})}/_{S_0([\omega]_{e'})} \supseteq {}^{S([\omega]_e)}/_{S_0([\omega]_e)} \text{ holds for every } e' \geq e \text{ and } e, e' \in E.$$

This yields a union of simple factor groups $\bigcup_{e\in E} {}^{S([\omega]_e)}/_{S_0([\omega]_e)} = {}^{M(E)}/_{S_0(\Omega)}$. Therefore ${}^{M(E)}/_{S_0(\Omega)}$ itself is simple. $M(E)$ is a normal subgroup of $S(E)$ minimal among the normal subgroups containing $S_0(E)$, i.e. that $M(E)$ does not show up in the normal system of the symmetric group on Ω.

Clearly, $F(E) \cap M(E) = B(E) \cap M(E) = S_0(\Omega)$, $B(E) \cdot M(E) \ntrianglelefteq F(E) \cdot M(E)$ and $F(E) \cdot M(E) \neq S(E)$, since $F(E) \not\subset S(E)$, if E is not finite. Again, from the Isomorphism Theorem we conclude the simplicity of the factors ${}^{B(E)\cdot M(E)}/_{B(E)}$ and ${}^{F(E)\cdot M(E)}/_{F(E)}$. Thus, this proves

Proposition 4.2 *$M(E)$ is a normal subgroup of $S(E)$ and the factor ${}^{M(E)}/_{S_0(\Omega)}$ is non-trivial simple if and only if E is not finite. In this case one gets:*

$$B(E) \cdot M(E) \trianglelefteq F(E) \cdot M(E) \trianglelefteq S(E).$$

The factors ${}^{B(E)\cdot M(E)}/_{B(E)}$ and ${}^{F(E)\cdot M(E)}/_{F(E)}$ are non-trivial simple.

Furthermore, let E have an approximating tower $T = (t_i)_{i\in\mathbb{N}}$. Recall that T is of the same type as E. Let $Alt(E) := \bigcup_{i\in\mathbb{N}} \prod_{\omega\in\Omega} Alt([\omega]_{t_i})$. In the case that E is finite it can be easily seen that $Alt(E)$ is a proper normal subgroup of $S(E) = F(E)$. What can be said about the cardinality of the non-trivial, elementary abelian factor group ${}^{S(E)}/_{Alt(E)}$? We shall define inductively a permutation $\pi \in S(E) \setminus Alt(E)$.

For $t_0 \in T$ we consider the partition of Ω in t_0-classes and, since Ω is countable,one has $\Omega = \bigcup_{i\in\mathbb{N}}[\omega]_{t_i}$. Suppose that every t_0-class contains at least two distinct elements ω_i, ω_i' such that a transposition $(\omega_i\omega_i')$ may be defined in $[\omega]_{t_0}$ for every $i \in \mathbb{N}$. Considering the first t_0-class $[\omega_1]_{t_0}$ let π_0 be the transposition $(\omega_1\omega_1')$. Now assume π_i has already been defined on $[\omega_1]_{t_i}$. To expand this permutation to $[\omega_1]_{t_{i+1}}$ the equivalence class of the coarser equivalence relation, suppose further without loss of generality that every t_{i+1}-class is union of at least three t_i-classes. Choose $[\omega_1]_{t_i}, [\omega_j]_{t_i}, [\omega_k]_{t_i}$ pairwise distinct t_i-classes contained in $[\omega_1]_{t_{i+1}}$. Define in $[\omega_j]_{t_i}$ and $[\omega_k]_{t_i}$ transpositions $(\omega_j\omega_j')$, respectively $(\omega_k\omega_k')$. Put π_{i+1} as ${}^{\pi_{i+1}}|_{[\omega_1]_{t_i}} = \pi_i$ and ${}^{\pi_{i+1}}|_{[\omega_1]_{t_{i+1}}\setminus[\omega_1]_{t_i}} = (\omega_j\omega_j')(\omega_k\omega_k')$. Hence π_{i+1} is odd. If one puts $\pi = \bigcup_{i\in\mathbb{N}} \pi_i$, then $\pi \in S(E)\setminus Alt(E)$ and for the cardinality of this factor one gets: $|{}^{S(E)}/_{Alt(E)}| \geq 2$. Moreover $\pi \notin S_0(\Omega) \cdot Alt(E)$ since there is an $e \in E$ such that π - by construction - is odd on infinitely many e-classes. Therefore one gets: $|{}^{S(E)}/_{Alt(E)}| \geq 4$.

Now let the set E of equivalence relations only be transitive and directed. Denote by $\tilde{S}_0(t_0) := S_0(t_0) = \prod_{\omega\in\Omega} S([\omega]_{t_0})$ and let $\tilde{S}_0(e') := \langle S_0(e'), \tilde{S}_0(e)\rangle^{S(e')}$ for $e' \geq e$. Put $\tilde{S}_0(E) := \bigcup_{e'\in E} \tilde{S}_0(e')$. $\tilde{S}_0(E)$ is a normal subgroup of $S(E)$ since for every $\pi \in S(E)$ there is an π-admissible e-class $e' \in E$ and therefore $\tilde{S}_0(e') \subseteq \tilde{S}_0(E)$ is π-invariant, for all $e' \geq e$. It can be easily seen that $\tilde{S}_0(E) = F(E)$ if E is finite. The same arguments prove that $\tilde{Alt}(E)$ is also a normal subgroup, independent of E being bounded, finite or infinite, for $\tilde{Alt}(E) := \bigcup_{e'\in E} \tilde{Alt}(e')$ with $\tilde{Alt}(t_0) := Alt(t_0) = \prod_{\omega\in\Omega} Alt([\omega]_{t_0})$ and $\tilde{Alt}(e') := \langle Alt(e'), \tilde{Alt}_0(e)\rangle^{Alt(e')}$ for $e' \geq e$. One gets the following

Proposition 4.3 *If the set E of equivalence relations is transitive and directed then the sets $\tilde{S}_0(E)$ and $\tilde{Alt}(E)$ are normal subgroups of $S(E)$. Further, $F(E)$ and $\tilde{S}_0(E)$, respectively $\tilde{Alt}(E)$ and $Alt(E)$ coincide if E is finite. Hence ${}^{S(E)}/_{S_0(\Omega)}$ is not simple.*

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