University of Mannheim
School of Social Sciences
Mathematics for Political Scientists, Fall 2023
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Solutions Basics and Set Theory

1. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8\}$ and $C = \{6, 8\}$. Find following:
   (a) $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$
   (b) $A \cap B = \{2, 4\}$
   (c) $A \cap B^C = \{1, 3, 5\}$
   (d) $B - A = \{6, 8\}$
   (e) $C - B = \emptyset$
   (f) $A \cap C = \emptyset$

2. Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4\}$ and $C = \{a, b, 1, 2\}$. Show that:
   (a) Distributivity: $(A \cap C) \cup (B \cap C) = (A \cup B) \cap C$

$$\begin{aligned} \{a, b\} \cup \{1, 2\} &= \{a, b, c, d, 1, 2, 3, 4\} \cap \{a, b, 1, 2\} \\ \{a, b, 1, 2\} &= \{a, b, 1, 2\} \end{aligned}$$

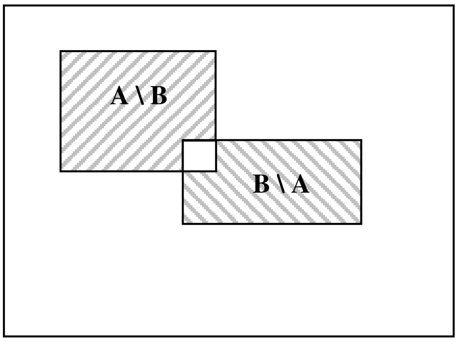
   (b) Associativity: $(A \cap B) \cap C = A \cap (B \cap C)$

$$\begin{aligned} \emptyset \cap \{a, b, 1, 2\} &= \{a, b, c, d\} \cap \{1, 2\} \\ \emptyset &= \emptyset \end{aligned}$$

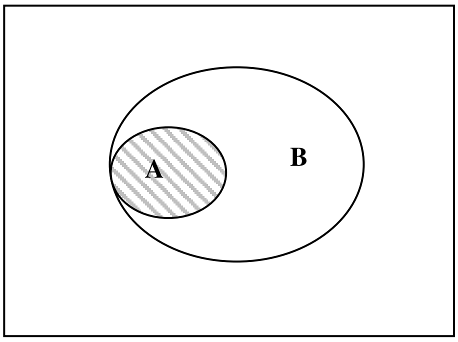
   (c) De Morgan Laws: $C - (A \cup B) = (C - A) \cap (C - B)$

$$\begin{aligned} \{a, b, 1, 2\} - \{a, b, c, d, 1, 2, 3, 4\} &= \{1, 2\} \cap \{a, b\} \\ \emptyset &= \emptyset \end{aligned}$$

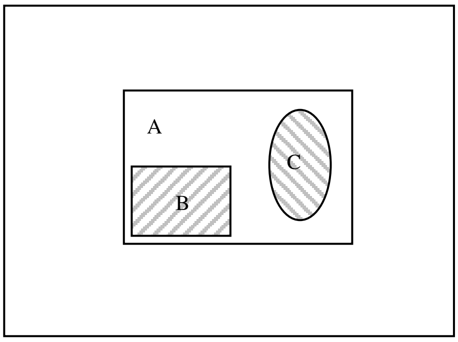
3. Determine which of the following formulas are true. If any formula is false, find a counterexample to demonstrate this using a Venn diagram.
   (a) $A \setminus B = B \setminus A$
   false



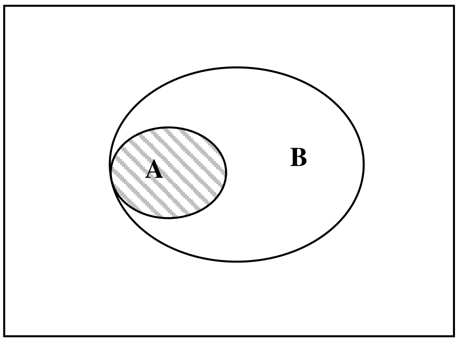
(b) $A \subseteq B \Longleftrightarrow A \cap B = A$
true



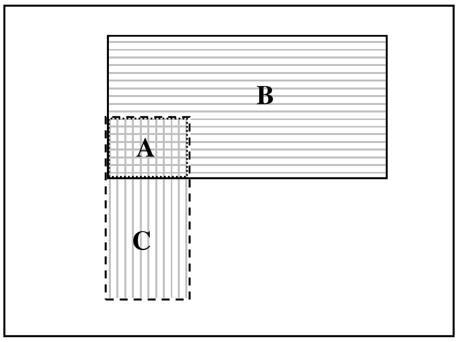
(c) $A \cup B = A \cup C \Longrightarrow B = C$
false



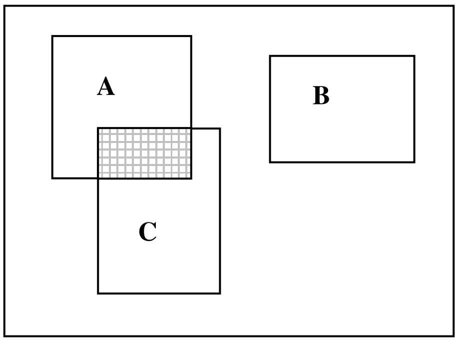
(d) $A \subseteq B \Longleftrightarrow A \cup B = B$
true



(e) $A \cap B = A \cap C \Longrightarrow B = C$
false



(f) $A \setminus (B \setminus C) = (A \setminus B) \setminus C$
false



4. Explain in words why it is true that for any sets A, B, C:

(a) $(A \cup B) \cup C = A \cup (B \cup C)$
This is true since the union of two sets contains all elements included in either set.

(b) $(A \cap B) \cap C = A \cap (B \cap C)$
This is true since an intersection only includes those elements that are included in both sets.

(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Let us think of B and C as a joint set. If we intersect this set with A, we receive $A \cap (B \cup C)$. If we now partition the joint set into two distinct sets and intersect these with A, we have partitioned $A \cap (B \cup C)$ into its two constituent elements $(A \cap B) \cup (A \cap C)$.

(d) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Since A is included in either bracket on the right-hand side of the equation, it is also included in their intersection. Thus, "factor it out" and form a union of it with the intersection of B and C.

5. Find the interior point(s) and the boundary points(s) of the set $\{x : 1 \leq x \leq 5\}$.

   (a) Interior points: $\{x : 1 < x < 5\}$

   (b) Boundary points: $\{x : x = 1 \vee x = 5\}$

6. Why does every set in $\mathbb{R}$ that is nonempty, closed, and bounded have a greatest member?

   Denoting such a set by S, $\sup S$ is a boundary point. Since S is closed, $\sup S \in S$ and so S has a greatest member.

7. Which of the following sets in $\mathbb{R}$ and $\mathbb{R}^2$ are open, closed, or neither?

   (a) $A = \{x \in \mathbb{R}^1 : x = 2 \text{ or } 3 < x < 4\}$ Neither since it contains one but not all of its boundary points.

   (b) In each of the following three cases, the boundary points are the points on the parabola $y = x^2$ with $-1 \leq x \leq 1$, and the points on the line $y = 1$ with $-1 \leq x \leq 1$.
   $B = \{(x, y) \in \mathbb{R}^2 : x^2 \leq y \leq 1\}$
   Closed since it contains all its boundary points.

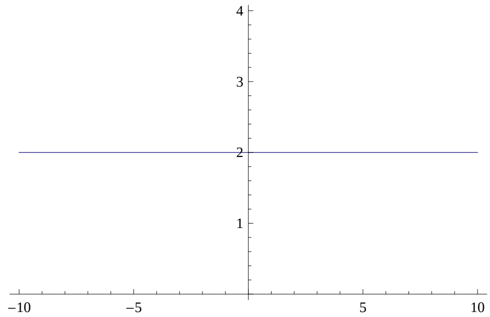
   (c) $C = \{(x, y) \in \mathbb{R}^2 : x^2 < y < 1\}$
   Open since it contains none of its boundary points.

   (d) $D = \{(x, y) \in \mathbb{R}^2 : x^2 \leq y < 1\}$
   Neither since it contains some but not all its boundary points.

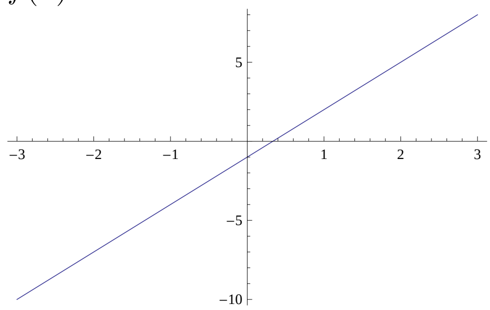
   (e) Universal set: both open and closed: "clopen".

8. Sketch the following functions:

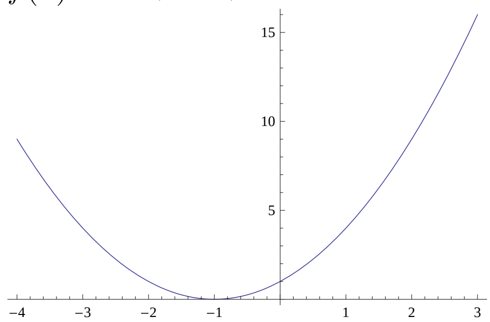
(a) $f(x) = 2$



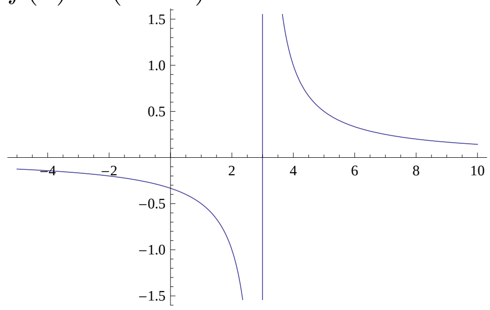
(b) $f(x) = 3x - 1$



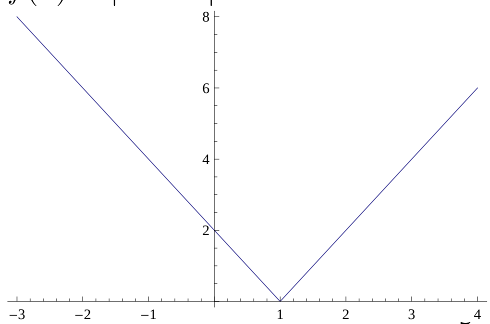
(c) $f(x) = x^2 + 2x + 1$



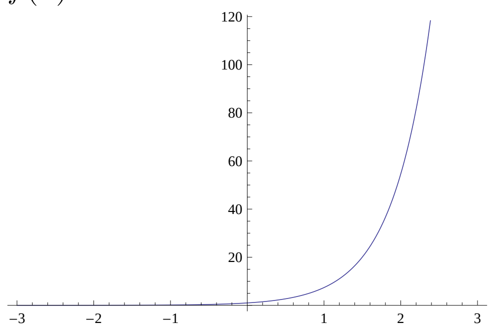
(d) $f(x) = (x - 3)^{-1}$



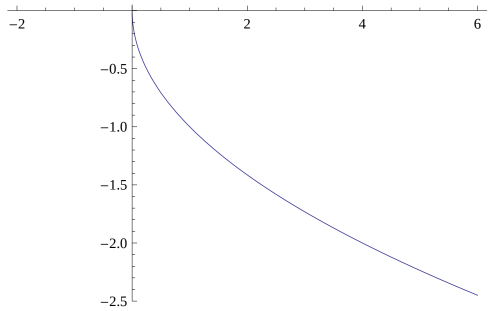
(e) $f(x) = |2x - 2|$



(f) $f(x) = e^{2x}$



(g) $f(x) = -\sqrt{x}$



9. Which of the following functions is injective, bijective, or surjective?

   (a) $a(x) = 2x + 1$
   $a(x)$ is both injective (every element of the domain is linked to at most one element in the co-domain) and surjective (since for every element in the co-domain there is at least one element in the domain) and, thus, bijective.

   (b) $b(x) = x^2$
   $b(x)$ is not injective since $b(x) = b(-x)$. It is also not surjective since there are no negative values for $b(x)$. However, if we would specify the range of $b(x) \in \mathbb{R}^+$, then it would be surjective.

   (c) $c(x) = \ln x$ for $(0, \infty) \mapsto \mathbb{R}$
   $c(x)$ is bijective.

   (d) $d(x) = e^x$ for $\mathbb{R} \mapsto \mathbb{R}$
   $d(x)$ is injective, but not surjective as there are no negative values for $d(x)$.

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Solutions Analysis I

1. Solve the following equations.

(a)

$$\begin{aligned} x^2 - 6x + 8 &= 0 \\ x_{1|2} &= \frac{6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 8}}{2 \cdot 1} \\ x_1 &= 4 \\ x_2 &= 2 \end{aligned}$$

(b)

$$\begin{aligned} (3x-1)^2 - (5x-3)^2 &= -(4x-2)^2 \\ 9x^2 - 6x + 1 - 25x^2 + 30x - 9 &= -16x^2 + 16x - 4 \\ 8x &= 4 \\ x &= 0.5 \end{aligned}$$

(c)

$$\begin{aligned} \sqrt{x^2 - 9} &= 9 - x \\ x^2 - 9 &= (9-x)^2 \\ x^2 - 9 &= x^2 - 18x + 81 \\ 18x &= 90 \\ x &= 5 \end{aligned}$$

(d)

$$\begin{aligned} \log_x(2x+8) &= 2 \\ 2x + 8 &= x^2 \\ x_{1|2} &= 1 \pm 3 \end{aligned}$$

(e)

$$\begin{aligned} e^{2x-5}+1 &= 4 \\ e^{2x-5} &= 3 \\ 2x-5 &= \ln 3 \\ x &= 0.5\ln 3 + 2.5 \end{aligned}$$

(f)

$$\begin{aligned} \log_2 \frac{2}{x} &= 3+\log_2 x \\ \frac{2}{x} &= 2^{3+\log_2 x} \\ \frac{2}{x} &= 2^3 \cdot 2^{\log_2 x} \\ \frac{2}{x} &= 8 \cdot x \\ x^2 &= 0.25 \\ x &= 0.5 \end{aligned}$$

(g)

$$\begin{aligned} (27)^{2x+1} &= \frac{1}{3} \\ 2x+1 &= \log_{27}\frac{1}{3} \\ 2x+1 &= -\frac{1}{3} \\ x &= -\frac{2}{3} \end{aligned}$$

2. Simplify the following expressions.

(a)

$$\begin{aligned} \frac{4^2\cdot 6^2}{3^3\cdot 2^3} &= \frac{(4\cdot 6)^2}{(3\cdot 2)^3} \\ &= \frac{24^2}{6^3} \\ &= \frac{1}{6}\cdot\frac{24^2}{6^2} \\ &= \frac{1}{6}4^2 \end{aligned}$$

(b)

$$\begin{aligned} \frac{(x+1)^3(x+1)^{-2}}{(x+1)^2(x+1)^{-3}} &= \frac{(x+1)^3\cdot(x+1)^3}{(x+1)^2\cdot(x+1)^2} \\ &= \frac{(x+1)^6}{(x+1)^4} \\ &= (x+1)^2 \end{aligned}$$

(c)

$$(-3xy^2)^3 = -27x^3y^6$$

(d)

$$\begin{aligned}
\frac{\frac{(x^2)^3}{x^4}}{\left(\frac{x^3}{(x^3)^2}\right)^{-2}} &= \frac{\frac{x^{2\cdot 3}}{x^4}}{\frac{x^{3\cdot(-2)}}{x^{3\cdot 2\cdot(-2)}}} \\
&= \frac{\frac{x^6}{x^4}}{\frac{x^{12}}{x^6}} \\
&= \frac{x^6}{x^4} \cdot \frac{x^6}{x^{12}} \\
&= \frac{1}{x^4}
\end{aligned}$$

(e)

$$\begin{aligned}
((2x+1)(2x-1))\,(4x^2+1) &= (4x^2 - 2x + 2x - 1)(4x^2+1) \\
&= (4x^2-1)(4x^2+1) \\
&= 16x^4 - 1
\end{aligned}$$

(f)

$$\begin{aligned}
\frac{6x^5 + 4x^3 - 1}{2x^2} &= \frac{6x^5 + 4x^3}{2x^2} - \frac{1}{2x^2} \\
&= 3x^3 + 2x - \frac{1}{2x^2}
\end{aligned}$$

(g)

$$\begin{aligned}
\frac{1 + 4x^2 + 6x}{2x-1} &= \frac{(4x^2+1) + 6x}{2x-1} \\
&= \frac{(2x+1)(2x-1) + 2 + 6x}{2x-1} \\
&= 2x + 1 + \frac{6x+2}{2x-1} \\
&= 2x + 1 + \frac{(6x-3)+5}{2x-1} \\
&= 2x + 1 + 3 + \frac{5}{2x-1} \\
&= 2x + 4 + \frac{5}{2x-1}
\end{aligned}$$

(h)

$$\begin{aligned}
\frac{x^2-5x+4}{x^2+2x-3}-\frac{x^2+2x}{x^2+5x+6} &= \frac{(x-1)(x-4)}{(x-1)(x+3)}-\frac{x(x+2)}{(x+2)(x+3)} \\
&= \frac{x-4}{x+3}-\frac{x}{x+3} \\
&= \frac{x-4-x}{x+3} \\
&= -\frac{4}{x+3}
\end{aligned}$$

3. Show that:

(a) $\sum_{i=1}^{N}(x_i-\mu_x)^2=\sum_{i=1}^{N}x_i^2-N\mu_x^2$. Hint: Note that $\mu_x=\frac{1}{N}\sum_{i=1}^{N}x_i$.

$$\begin{aligned}
\sum_{i=1}^{N}(x_i-\mu_x)^2 &= \sum_{i=1}^{N}(x_i^2-2\mu_x x_i+\mu_x^2) \\
&= \sum_{i=1}^{N}x_i^2-2\mu_x\sum_{i=1}^{N}x_i+\sum_{i=1}^{N}\mu_x^2 \\
&= \sum_{i=1}^{N}x_i^2-2\mu_x N\mu_x+N\mu_x^2 \\
&= \sum_{i=1}^{N}x_i^2-N\mu_x^2.
\end{aligned}$$

(b) $\sum_{i=1}^{n}(a_{i+1}-a_i)=a_{n+1}-a_1$.

$$\begin{aligned}
\sum_{i=1}^{N}(a_{i+1}-a_i) &= \sum_{i=1}^{N}a_{i+1}-\sum_{i=1}^{N}a_i \\
&= \sum_{i=1}^{N}a_i+a_{N+1}-a_1-\sum_{i=1}^{N}a_i \\
&= a_{N+1}-a_1.
\end{aligned}$$

4. Show that $a^3-b^3=(a-b)(a^2+ab+b^2)$

$$\begin{aligned}
a^3-b^3 &= a(a^2-b^2)+ab^2-b^3 \\
&= a(a+b)(a-b)+b^2(a-b) \\
&= (a-b)(a^2+ab+b^2)
\end{aligned}$$

5. Differentiate the following functions with respect to x.

(a) $f(x)=7x^3-2x^2-5x+1$
$f'(x)=21x^2-4x-5$

(b) $f(x) = 0.7x^{-4} + 1.3 - 3.1x^3$
$f'(x) = -2.8x^{-5} - 9.3x^2$

(c) $f(x) = \dfrac{3x^2+1}{2x}$
$f'(x) = \dfrac{3}{2} - \dfrac{1}{2x^2}$

(d) $f(x) = \sqrt{4x+9}$
$f'(x) = \dfrac{2}{\sqrt{4x+9}}$

(e) $f(x) = \dfrac{x^{\frac{1}{3}} - 2}{(x^5-2)^3}$
$f'(x) = \dfrac{\frac{1}{3}x^{-\frac{2}{3}} \cdot (x^5-2)^3 - (x^{\frac{1}{3}} - 2) \cdot 3(x^5-2)^2 \cdot 5x^4}{(x^5-2)^6}$

(f) $f(x) = \ln\left(\dfrac{x^2}{x^4+1}\right)$
$f'(x) = \dfrac{x^4+1}{x^2} \cdot \dfrac{2x(x^4+1) - x^2 \cdot 4x^3}{(x^4+1)^2} = \dfrac{2}{x} - \dfrac{4x^3}{x^4+1}$

(g) $f(x) = e^{x^3+x}$
$f'(x) = e^{x^3+x} \cdot (3x^2+1)$

(h) $f(x) = \dfrac{1}{e^x + e^{-x}}$
$f'(x) = -1(e^x + e^{-x})^{-2} \cdot (e^x - e^{-x}) = -\dfrac{e^x - e^{-x}}{(e^x+e^{-x})^2}$

6. Find the all first and second (mixed) partial derivatives of the following functions.

(a) $f(x,y) = \ln x \cdot y^2$

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{y^2}{x} \\
\frac{\partial^2 f}{\partial x^2} &= -\frac{y^2}{x^2} \\
\frac{\partial f}{\partial y} &= 2y \ln x \\
\frac{\partial^2 f}{\partial y^2} &= 2 \ln x \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{2y}{x}
\end{aligned}
$$

(b) $f(x,y) = \sqrt{2x - y}$

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= (2x-y)^{-\frac{1}{2}} \\
\frac{\partial^2 f}{\partial x^2} &= -(2x-y)^{-\frac{3}{2}} \\
\frac{\partial f}{\partial y} &= -\frac{1}{2}(2x-y)^{-\frac{1}{2}} \\
\frac{\partial^2 f}{\partial y^2} &= -\frac{1}{4}(2x-y)^{-\frac{3}{2}} \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{1}{2}(2x-y)^{-\frac{3}{2}}
\end{aligned}
$$

(c) $f(x,y) = (x+4y)(e^{-2x} + e^{-3y})$

$$\begin{aligned}
f(x,y) &= (x+4y)(e^{-2x}+e^{-3y}) \\
&= xe^{-2x} + xe^{-3y} + 4ye^{-2x} + 4ye^{-3y} \\
\frac{\partial f}{\partial x} &= e^{-2x} + x\cdot(-2)\cdot e^{-2x} + e^{-3y} - 8ye^{-2x} \\
&= (1-2x-8y)e^{-2x} + e^{-3y} \\
\frac{\partial^2 f}{\partial x^2} &= -2(1-2x-8y)e^{-2x} - 2e^{-2x} \\
&= 4(-1+x+4y)e^{-2x} \\
\frac{\partial f}{\partial y} &= -yxe^{-3y} + 4e^{-2x} + 4e^{-3y} - 12ye^{-3y} \\
&= (4-xy-12y)e^{-3y} + 4e^{-2x} \\
\frac{\partial^2 f}{\partial y^2} &= -3(4-xy-12)e^{-3y} - 12e^{-3y} \\
&= 3(-8+xy+12)e^{-3y} \\
\frac{\partial^2 f}{\partial x \partial y} &= -8e^{-2x} - 3e^{-3y}
\end{aligned}$$

7. For what value of a is the following function continuous for all x? Is it also differentiable for all x for this value of a?

$$f(x) = \begin{cases} ax-1 & \text{if } x \le 1 \\ 3x^2+1 & \text{if } x > 1 \end{cases}$$

Take the limit of the (sub)function $3x^2+1$ at $x_0 = 1$

$$\begin{aligned}
f(x_0+h) &= \lim_{h\to 0} 3(x_0+h)^2 + 1 \\
f(1+h) &= 3\lim_{h\to 0}(1+h)^2 + 1 \\
&= 4
\end{aligned}$$

To have continuity, both (sub)functions have to have the same output in the limit. Hence, we equate 4 with the $a-1$. We receive $a = 5$.

To check for differentiability, we compute the derivative of $g(x) = 5x-1$, which is $g'(x) = 5$ and $g'(1) = 5$. Since 1 is not in the domain of $h(x) = 3x^2+1$, we have to take the limit:

$$\begin{aligned}
h'(x_0) &= \lim_{h\to 0} \frac{(3(x_0+h)^2+1)-(3x_0^2+1)}{h} \\
h'(1) &= \lim_{h\to 0} \frac{(3(1+h)^2+1)-(3+1)}{h} \\
&= \lim_{h\to 0} \frac{3(1+2h+h^2)+1-3-1}{h} \\
&= \lim_{h\to 0} \frac{6h+3h^2}{h} \\
&= \lim_{h\to 0} 6+3h \\
&= 6
\end{aligned}$$

Since the derivatives at $x_0 = 1$ are not the same, the function can not be differentiable at $x_0 = 1$.

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Solutions Analysis II

1. Suppose the function f is defined for all $x \in [-1.5, 2.5]$ by $f(x) = x^5 - 5x^3$.

   (a) Determine for which values of x the value of the function is equal to zero.

$$\begin{aligned} x^5 - 5x^3 &= 0 \\ x^5 &= 5x^3 \\ x^2 &= 5 \\ x &= \pm\sqrt{5} \end{aligned}$$

   From the second equation we see that $x = 0$ is a possible solution. For $x = \pm\sqrt{5}$ we have to check whether these points are in our domain. This is true for $x = \sqrt{5}$, but not for $x = -\sqrt{5}$. Thus, the function has two roots.

   (b) Calculate $f'(x)$ and find the extreme points of f. What is the maximum/the minimum of the function.
   $f'(x) = 5x^4 - 15x^2$. The FOC gives us.

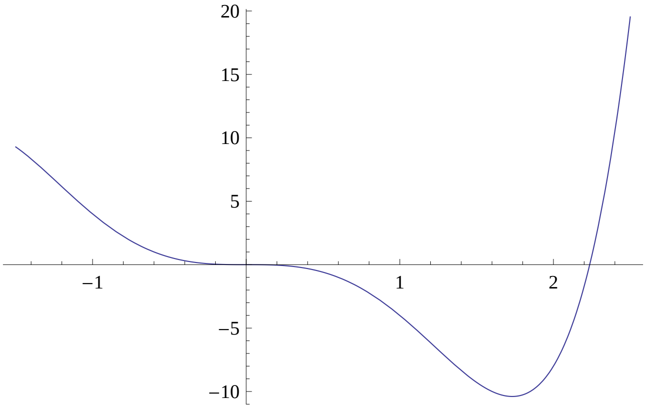
$$\begin{aligned} 5x^4 - 15x^2 &= 0 \\ 5x^4 &= 15x^2 \\ x^2 &= 3 \\ x &= \pm\sqrt{3} \end{aligned}$$

   When checking for the domain, we find that $x = 0$ and $x = \sqrt{3}$ serve as possible extreme points. Now we need to check the SOC.

$$\begin{aligned} f''(x) &= 20x^3 - 30x \\ f''(x = 0) &= 0 \\ f''(x = \sqrt{3}) &= 30\sqrt{3} > 0 \end{aligned}$$

   At $x = 0$ we have a saddle point. At $x = \sqrt{3}$ there is a minimum.
   Are there any other minima/maxima? Yes, of course. We have to consider the boundaries of our domain. Both at $x = -1.5$ and $x = 2.5$ we have additional maxima.
   The overall maximum of the function is attained at $x = 2.5$ with $f(x) \approx 19.5$. The overall minimum is $x = \sqrt{3}$ with $f(x) \approx -10.4$.

(c) Does the function have inflection points?
Yes, it does. We already found the first inflection point, which also happens to be a saddle point.
We find the additional inflection points by setting $f''(x) = 0$. This gives $x = \pm\sqrt{1.5}$.



2. Which of the following functions of x are convex? Which are concave?

(a) $f(x) = (2x - 1)^6$
$f'(x) = 6(2x - 1)^5 \cdot 2$
$f''(x) = 5 \cdot 12(2x - 1)^4 \cdot 2 \geq 0 \Longrightarrow$ convex

(b) $f(x) = 5x + 7$
The function is both convex and concave since the sets of points above and below the function are convex.

(c) $f(x) = x^5$
$f'(x) = 5x^4$
$f''(x) = 20x^3$
The function as a whole is neither convex nor concave (but we can specify this for parts of the function).

(d) $f(x) = \sqrt{1 + x^2}$
$f'(x) = x(1 + x^2)^{-\frac{1}{2}}$
$f''(x) = (1 + x^2)^{-\frac{1}{2}} + x^2(1 + x^2)^{-\frac{3}{2}} > 0 \Longrightarrow$ strictly convex

(e) $f(x) = x^5$ for $x \geq 0$
$f''(x) = 20x^3 \geq 0 \; \forall \; x \geq 0 \Longrightarrow$ convex

(f) $f(x) = 5x^2 - x^4$ for $x \geq 1$
$f'(x) = 10x - 4x^3$
$f''(x) = 10 - 12x^2 < 0 \; \forall \; x \geq 1 \Longrightarrow$ strictly concave

3. Appeasement Problem (Ashworth and Bueno de Mesquita, 2006). For full text see exercise set.

(a) Take the derivative with respect to x, set up the FOC, and solve for x.

$$
\begin{aligned}
1 - 2x - q &= 0 \\
\underset{2}{x^*(q)} &= \frac{1-q}{2}
\end{aligned}
$$

$x^*(q)$ represents state S's optimal choice of appeasement as a function of S's perceived military strength.

(b) We can find comparative statics by examining how this equilibrium offer ($x^*(q)$) changes when q changes. Differentiating $x^*(q)$ with respect to q yields:

$$\frac{\partial x^*(q}{\partial q} = -\frac{1}{2} < 0$$

Not surprisingly, the optimal offer is decreasing in q. The stronger S is militarily, the less willing S is to appease D.

4. Consider the function $f(x) = (x^2 + 2x)e^{-x}$.

(a) Determine for which values of x the value of the function is equal to zero. We have to set $(x^2 + 2x)e^{-x} = 0$. We know that $e^{-x} > 0 \; \forall \; x \in \mathbb{R}$. Thus,

$$\begin{aligned} x^2 + 2x &= 0 \\ x &= \frac{-2 \pm \sqrt{4-0}}{2} = -1 \pm 1 \end{aligned}$$

The roots of the function are $x = -2$ and $x = 0$.

(b) Calculate f'(x) and find the extreme points of f. What is the maximum/the minimum of the function?

$$\begin{aligned} f(x) &= (x^2 + 2x)e^{-x} \\ f'(x) &= -(x^2 + 2x)e^{-x} + (2x + 2)e^{-x} \\ f''(x) &= (x^2 + 2x)e^{-x} - 2(2x + 2)e^{-x} + 2e^{-x} \end{aligned}$$

We take the FOC $f'(x) = 0$ to look for stationary points.

$$\begin{aligned} -(x^2 + 2x)e^{-x} + (2x + 2)e^{-x} &= 0 \\ -(x^2 + 2x) + (2x + 2) &= 0 \\ -x^2 + 2 &= 0 \\ x &= \pm\sqrt{2} \end{aligned}$$

We have stationary points at $x = \pm\sqrt{2}$. We now have to check the SOC.

$$\begin{aligned} f''(x) &= (x^2 + 2x)e^{-x} - 2(2x + 2)e^{-x} + 2e^{-x} \\ &= (x^2 + 2x - 2x - 2 - 2x - 2 + 2)e^{-x} \\ &= (x^2 - 2x - 2)e^{-x} \end{aligned}$$

Again, we know that $e^{-x} > 0 \; \forall \; x \in \mathbb{R}$, so that we only have to consider the polynomial. For $x = -\sqrt{2}$, the polynomial $(2 + 2\sqrt{2} - 2) > 0 \Longrightarrow$ local minimum.

For $x = \sqrt{2}$, the polynomial $(2 - 2\sqrt{2} - 2) < 0 \Longrightarrow$ local maximum.
As the domain is not limited, we have to check for the limit of $f(x)$ for $x \longrightarrow \pm\infty$ in order to specify whether the local extreme points are also global.

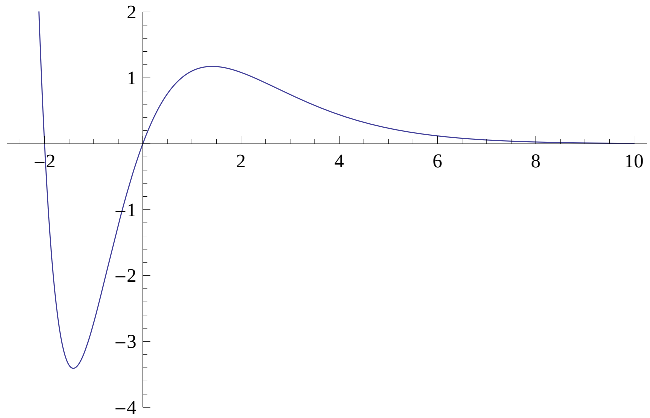
$$\begin{aligned}\lim_{x\to-\infty}(x^2+2x)e^{-x} &\approx e^{-x} = \infty\\ \lim_{x\to\infty}(x^2+2x)e^{-x} &\approx e^{-x} = 0\end{aligned}$$

Therefore, the function does not have a global maximum. However, it has a global minimum since $f(-\sqrt{2}) < \lim_{x\to\infty} f(x)$.

(c) Does the function have inflection points?
Yes, the function does have inflection points.

$$\begin{aligned}f''(x) &= 0\\ (x^2+2x)e^{-x} - 2(2x+2)e^{-x} + 2e^{-x} &= 0\\ (x^2+2x) - 2(2x+2) + 2 &= 0\\ x^2 - 2x - 2 &= 0\\ x &= \frac{2 \pm \sqrt{4+8}}{2}\\ x &= 1 \pm \sqrt{3}\end{aligned}$$

(d) Sketch the function and specify whether it is convex/concave (in sections).



The function is neither convex nor concave as a whole.
For concavity/convexity in parts of the function the inflection points are crucial.
We can see from the graph that the function is convex for all $x \in (-\infty, 1-\sqrt{3}]$ and $x \in [1+\sqrt{3}, \infty)$.
It is concave for all $x \in [1-\sqrt{3}, 1+\sqrt{3}]$.

5. Derivate the indefinite integrals:

(a) $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + C$

(b) $\int e^{-4t} dt = -\dfrac{1}{4e^{4t}} + C$

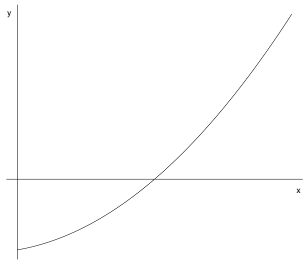
(c) $\int x\sqrt{x} dx = \frac{2}{5} x^{\frac{5}{2}} + C$

(d) $\int \frac{1}{x} = \ln x dx + C$

(e) $\int (2x^2 + x - 3) dx = \frac{2}{3}x^3 + \frac{1}{2}x^2 - 3x + C$

(f) $\int \frac{(x^4+1)^2}{x^3} dx = \frac{1}{6}x^6 + x^2 - \frac{1}{2x^2} + C$

6. Calculate $\int_0^2 (2x^2 + x - 3)dx$. Hint: Make a sketch of the function before.



At $x = 1$, the curve crosses the horizontal axis. In order to compute the total area "under the curve", compute $\int_0^1 (2x^2 + x - 3)dx + \int_1^2 (2x^2 + x - 3)dx$. Using the result from 1e, we can write:

$$\begin{aligned}\int_0^1 (2x^2 + x - 3)dx + \int_1^2 (2x^2 + x - 3)dx =& \left|\frac{2}{3} + \frac{1}{2} - 3 - (0)\right| + \\ & \left|\frac{2}{3}2^3 + \frac{1}{2}2^2 - 3 \cdot 2 - (\frac{2}{3} + \frac{1}{2} - 3)\right| \\ =& \frac{11}{6} + \frac{4}{3} + \frac{11}{6} \\ =& 5\end{aligned}$$

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Solutions: Linear Algebra

1. Consider the following matrices and vectors.

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 8 & 3 \\ 0 & -1 & 6 \end{pmatrix} ; \ \mathbf{B} = \begin{pmatrix} -3 & 2 & 4 \\ 2 & 3 & 4 \\ 2 & -4 & 0 \end{pmatrix} ; \ \mathbf{c} = \begin{pmatrix} 4 & -3 & 2 \end{pmatrix} ; \ \mathbf{d} = \begin{pmatrix} 3 & 8 \end{pmatrix} ;$$

$$\mathbf{e} = \begin{pmatrix} 2 & 6 & 9 \end{pmatrix} ; \ \mathbf{F} = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} ; \ \mathbf{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ; \ \mathbf{H} = \begin{pmatrix} 5 & 6 & 1 \\ -2 & 7 & 8 \end{pmatrix} ;$$

$$\mathbf{K} = \begin{pmatrix} a_1 & ... & a_n \\ b_1 & ... & b_n \end{pmatrix}$$

Do the calculations if possible.

(a) $\mathbf{M}_1 = \mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} 13 & -9 & 16 \\ 16 & 16 & 40 \\ 10 & -27 & -4 \end{pmatrix}$

(b) $\mathbf{M}_2 = \mathbf{A} - \mathbf{B} = \begin{pmatrix} 4 & 1 & 1 \\ 0 & 5 & -1 \\ -2 & 3 & 6 \end{pmatrix}$

(c) $\mathbf{M}_3 = \mathbf{B} \cdot \mathbf{F} = n.d.$
Not possible since ncol($\mathbf{B}$)$\neq$nrow($\mathbf{F}$).

(d) $\mathbf{M}_4 = \mathbf{A} \cdot \mathbf{c} = n.d.$
Not possible since ncol($\mathbf{A}$)$\neq$nrow($\mathbf{c}$)

(e) $\mathbf{M}_5 = \mathbf{c} \cdot \mathbf{A} = \begin{pmatrix} -2 & -14 & 23 \end{pmatrix}$

(f) $\mathbf{m}_6 = \mathbf{d} \cdot \mathbf{c} = n.d.$
Not possible since the vectors have different dimension.

(g) $\mathbf{m}_7 = 2\mathbf{c} \cdot 3\mathbf{e} = 48.$

(h) $\mathbf{M}_8 = \mathbf{B} \cdot \mathbf{G} = \mathbf{B}$

(i) $\mathbf{M}_9 = \mathbf{A} \cdot \mathbf{H} = n.d.$
Not possible since ncol($\mathbf{A}$)$\neq$ nrow($\mathbf{H}$)

(j) $\mathbf{M}_{10} = \mathbf{H}' \cdot \mathbf{F} = \begin{pmatrix} 13 & -4 \\ 25 & 14 \\ 11 & 16 \end{pmatrix}$

2. What is the dimension of the following matrices?

   (a) $\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{H}'$ 3×2

   (b) $\mathbf{c} + \mathbf{e} \cdot \mathbf{H}'$ not possible, since $\text{ncol}(\mathbf{c}) = 3$ and $\text{ncol}(\mathbf{e} \cdot \mathbf{H}') = 2$

   (c) $\mathbf{F} \cdot \mathbf{K}$ $2 \times n$

3. Specify whether the following matrices are square, zero, identity, diagonal, or upper/lower triangular matrices and give their dimension as well as their rank.

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 5 & 0 & 8 \\ 0 & 1 & -2 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 7 & 0 \\ 1 & -3 & 9 \end{pmatrix}, \mathbf{E} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 8 & 0 \\ 0 & -5 & 0 \end{pmatrix}$$

	A	**B**	**C**	**D**	**E**
dimension	(3×2)	(2×2)	(2×3)	(3×3)	(3×3)
rank	0	2	2	3	2
square matrix		x		x	x
zero matrix	x				
identity matrix		x			
diagonal matrix		x			
upper triangular matrix	(x)	x	(x)		
lower triangular matrix	(x)	x			x

4. Is the equation $(\mathbf{F} + \mathbf{G})^2 = \mathbf{F}^2 + 2 \cdot \mathbf{F} \cdot \mathbf{G} + \mathbf{G}^2$ true for any square matrices of the same order?

   $(\mathbf{F} + \mathbf{G})^2 = (\mathbf{F} + \mathbf{G}) \cdot (\mathbf{F} + \mathbf{G}) = \mathbf{F}^2 + \mathbf{F} \cdot \mathbf{G} + \mathbf{G} \cdot \mathbf{F} + \mathbf{G}^2$
   No, this is only the case if $\mathbf{F} \cdot \mathbf{G} = \mathbf{G} \cdot \mathbf{F}$.

5. Find all 2×2 matrices $\mathbf{A}$ such that $\mathbf{A}^2$ is the matrix obtained from $\mathbf{A}$ by squaring each entry.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} aa + bc & ab + bd \\ ac + cd & bc + dd \end{pmatrix} \equiv \begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix}$$

   From the identity we derive the following system of equations.

$$aa + bc = aa \quad (1)$$
$$ab + bd = bb \quad (2)$$
$$ac + cd = cc \quad (3)$$
$$bc + dd = dd \quad (4)$$

From (1) and (4) we know that $bc = 0$. The first and easy solution, thus, is all matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

A little bit more subtle are the other two possible solutions

$$\begin{pmatrix} a & 0 \\ a+d & d \end{pmatrix} \text{ and } \begin{pmatrix} a & a+d \\ 0 & d \end{pmatrix}.$$

For the proof, note that either b or c has to be equal to zero. For symmetry we focus on the case where $b = 0$. Let $c = a + d$, then equation (3)

$$\begin{aligned} ac + cd &= cc \\ a(a+d) + d(a+d) &= (a+d)(a+d) \\ a^2 + 2ad + d^2 &= (a+d)^2. \end{aligned}$$

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Solutions: Probability Theory

1. Suppose we draw 5 balls from an urn containing 15. How many different sets of drawn balls are there? Consider permutation and combination both with and without repetition.

   Ordered, with repetition: $n^k = 15^5 = 759,375$
   Ordered, without repetition: $\binom{n}{k}k! = \frac{n!}{k!(n-k)!}k! = \frac{15!}{10!} = 360,360$
   Unordered, with repetition: $\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!} = 11,628$
   Unordered, without repetition: $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{15!}{5!(15-5)!} = 3,003$

2. Make a complete list of all the different subsets of the set $\{a, b, c\}$. How many are there if the empty set and the set itself are included? Do the same for the set $\{a, b, c, d\}$. In addition: How many different subsets are there for a set containing n elements?

   Subsets of $\{a, b, c\}$: $\{a\}$, $\{b\}$, $\{c\}$, $\{ab\}$, $\{ac\}$, $\{bc\}$, $\{abc\}$, $\emptyset$, that is 8 subsets.
   Subsets of $\{a, b, c, d\}$: $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{ab\}$, $\{ac\}$, $\{ad\}$, $\{bc\}$, $\{bd\}$, $\{cd\}$, $\{abc\}$, $\{abd\}$, $\{acd\}$, $\{bcd\}$, $\{abcd\}$, $\emptyset$, which is 16 subsets overall.

   Let us think of the possible subsets of size k of a set of size n. We can calculate their number as a combination without repetition. For example, if we are interested in a subset with 4 elements of a set with 10 elements, there are $\binom{10}{4}$ possible combinations. The number of all possible subsets of a set containing n elements is, thus, given by $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

3. The World Health Organization (WHO) recently detected a new, deadly disease. This so called "CAT-Virus" is currently making its way to Europe and the US. The first symptoms are hallucination and dizziness. Quickly followed by pain, fever and shock. Death might follow within 2-4 days. The WHO developed a quick test that everyone can self-administer. The test is 99% accurate. That is, if you have the disease, there is a 99 percent chance that the test will detect it. If you don't have the disease, the test will be 99 percent accurate in saying that you don't. In the general population, 0.1% (that is a tenth of a percentage point) of the people have

the disease. When you get a positive test result, what is the probability that you actually have the CAT-Virus?

$$
\begin{aligned}
P(A) &= 0.01/100 \\
P(A^c) &= 1 - 0.01/100 \\
P(B|A) &= 0.99 \\
P(B|A^c) &= 1 - 0.99
\end{aligned} \tag{1}
$$

Bayes Theorem says:

$$
P(A|B) = \frac{P(B|A)P(A)}{P(B)} \tag{2}
$$

Law of Total Probability says:

$$
P(B) = P(A)P(B|A) + P(A^c)P(B|A^c) \tag{3}
$$

Substituting and computing:

$$
\begin{aligned}
P(A|B) &= \frac{P(B|A)P(A)}{P(A)P(B|A) + P(A^c)P(B|A^c)} \\
&= \frac{0.01/100 \cdot 0.99}{0.01/100 \cdot 0.99 + (1 - 0.01/100) \cdot (1 - 0.99)} \\
&= 0.09
\end{aligned} \tag{4}
$$

4. Consider the p.d.f. $f(x) = 2x$ for $0 \le x \le 1$.
   (a) Calculate the c.d.f. of $f(x)$.

   $$F(x) = x^2$$

   (b) Is $f(x)$ a proper p.d.f.?

   $$
   \begin{aligned}
   \int_0^1 F(x) &= 1 \\
   1^2 - 0^2 &= 1
   \end{aligned}
   $$

   $f(x)$ is a proper p.d.f.

5. Consider the c.d.f. $G(x) = \frac{1}{9}x^2$ for $0 \le x \le 3$.

(a) Calculate the p.d.f. of $G(x)$, $g(x)$.

$$g(x) = \frac{2}{9}x$$

(b) Is $g(x)$ a proper p.d.f.?

$$\begin{aligned}\int_0^3 G(x) &= 1 \\ \frac{1}{9} \times 3^2 - \frac{1}{9} \times 0^2 &= 1\end{aligned}$$

$g(x)$ is a proper p.d.f.

6. Consider the p.d.f. $h(x) = \frac{4}{3}(1 - x^3)$ for $0 < x < 1$. Determine

(a) $\Pr(X < \frac{1}{2})$.

$$H(x) = -\frac{1}{3}x(x^3 - 4)$$
$$H\left(\frac{1}{2}\right) \approx 0.65$$

(b) $\Pr(X > \frac{1}{3})$.

$$1 - H\left(\frac{1}{3}\right) \approx 0.56$$

(c) $\Pr(\frac{1}{4} < X < \frac{3}{4})$.

$$H\left(\frac{3}{4}\right) - H\left(\frac{1}{4}\right) \approx 0.56$$

7. Consider the p.d.f. $k(x) = cx^2$ for $1 \le x \le 2$. Determine

(a) Find the value of the constant c.

$$\begin{aligned}\int_1^2 k(x)dx &= 1 \\ \frac{1}{3}cx^3\Big|_1^2 &= 1 \\ c &= \frac{3}{7}\end{aligned}$$

(b) Find $\Pr(X > \frac{3}{2})$.

$$K(x) = \frac{1}{7}x^3$$
$$1 - K\left(\frac{3}{2}\right) \approx 0.52$$