Mathematics for Political Scientists

Master

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Fall 2023

Introduction

Course Objectives

What is this course about?

- This course is designed for self-study
- Recap of your high-school / Abitur knowledge in mathematics.
- Introduction to the fundamentals in math that are necessary for your understanding of statistics and game theory.
- Overcome possible reservations against the use of mathematics.
- A refresher and starting point for future individual learning.

What is this course not about?

- It is not a mathematical freak show!
- It does not introduce advanced mathematical techniques.

Course Objectives

Instructions

1. Check the accompanying syllabus
2. Work through the slides
3. If you are not familiar with one of the topics and/or feel like you need more detailed information to understand the material:
    - Read chapters from the recommended books in the syllabus
    - Watch video tutorials suggested in the syllabus
4. Work through the exercise sheets
5. Check your results with the solution sheets
6. In case of questions or feedback on the material contact cgueiros@uni-mannheim.de

Course Objectives

Why is math important to social scientists?

- Mathematics allows for orderly and systematic communication. Ideas expressed mathematically can be more carefully defined and more directly communicated than narrative language, which is susceptible to vagueness and misinterpretation.
- Mathematics is an effective way to describe and model our world.

Applications

- Game Theory, Decision Theory
- Computer Simulation, Agent-Based Modeling
- Statistics, Econometrics
- Empirical Analyses in any field

Course Objectives

Mathematical confidence: Many students of mathematics are hindered by false beliefs about the subject and/ or themselves. Here are some things to keep in mind if you find mathematics daunting:

- Every person is capable of doing mathematics.
- Being good at mathematics doesn't mean being fast at mathematics.
- If you believe that you can learn, you will learn more.
- If you struggled maths at school, you aren't doomed to struggle forever.
- Mathematics is learned by doing, not reading/ listening. It's essential to try. Mistakes are good for your brain.

Syllabus

I. Set Theory (The Basics)
   - introduction, functions

II. Analysis/Calculus
   - derivatives, optimization, integration

III. Linear Algebra
   - vectors, matrices

IV. Probability Theory
   - combinatorics, conditional probabilities, distributions

General Readings

Recommended:

General

- Gill (2006): Essential Mathematics for Political and Social Research.
- Moore/Siegel (2013): A Mathematics Course for Political and Social Research. *An introductory mathematics course aimed at social scientists, provides good intuitions for basic concepts and applications. It has accompanying video lectures on Youtube.*
- Simon/Blume (1994) *A comprehensive treatment of mathematics for students of economics for both undergraduate and more advanced level.*
- Sydsaeter/Hammond (2008) *Another standard mathematics textbook for economics undergraduates.*

Specific Readings

- Calculus
  - Spivak (2006) *A classic standard textbook for a first class in Calculus for mathematics students at undergraduate level.*
- Probability Theory
  - DeGroot/Schervish (2011) *A comprehensive standard treatment of probability and statistics for mathematics undergraduate students. Intuitive and (relatively) rigorous at the same time with lots of exercises.*
- Linear Algebra
  - Lay (2011) *A standard introduction for mathematics undergraduates.*
  - The Matrix Cookbook[1]
    An overview over some more advanced matrix calculus.

[1] http://www2.imm.dtu.dk/pubdb/views/edoc_download.php/3274/pdf/imm3274.pdf

Set Theory

Set Theory (The Basics)

Resources:

- Moore/Siegel: Chapter 1
- Gill: Chapter 1

Motivation

Explanations of political outcomes often begin with the presumption that such outcomes are the result of purposive decisions made by relevant individuals (e.g. voters, legislators) or groups of individuals (e.g. political parties, interest groups, nation states)

Fundamental to these kind of explanations are the concepts of ‘choice’ and ‘preferences’.

Set Theory is fundamental to the formalization of these concepts.

Set Theory is fundamental to the understanding of many other fields of mathematics, e.g. the concept of ‘functions’.

What Is a Set?

Definition (Set)

A **set** is a collection of distinct objects, where the objects therein are called **elements** or **members**.

For example $A = \{1, 2, 3\}$ is a set, and 1 is an element of A (write $1 \in A$), whereas 4 is not an element of A ($4 \notin A$).

If a set does not contain any elements, we call it an **empty set**. The shorthand for an empty set is $\emptyset$ or $\{\}$.

Example: Sets of Numbers

Symbol	Explanation	Example
$\mathbb{N}$	set of natural numbers	$1, 2, 3, 4, \ldots$
$\mathbb{Z}$	set of integers	$-2, -1, 0, 1, 2, \ldots$
$\mathbb{Q}$	set of rational numbers (fractions)	$-\frac{9}{7}, -1, 0, \frac{1}{2}, 1, \ldots$
$\mathbb{R}$	set of real numbers	fractions plus e.g. π or e
$\mathbb{R}^+$	set of positive real numbers	
$\mathbb{C}$	set of complex numbers	$\sqrt{-1}$

Relations of Sets

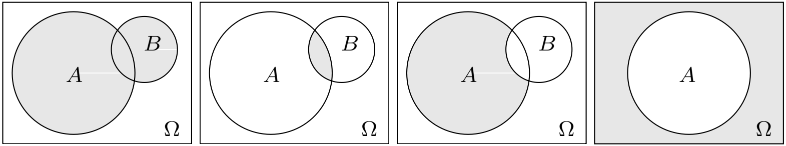
A set itself can, furthermore, be part of another set. E.g. $A = \{1, 2, 3\}$ is part of $B = \{1, 2, 3, 4\}$. We then say that A is a **subset** of B and write $A \subseteq B$. In particular it is true for every set A that $A \subseteq A$.

If A is a subset of B, but not equal to B (like in the example above), we call A a **proper** or **strict subset** of B and write $A \subset B$.

If two sets do not have any element in common, these sets are said to be **disjoint**. E.g. $A = \{1, 2, 3\}$ and $C = \{4, 5\}$ are disjoint.

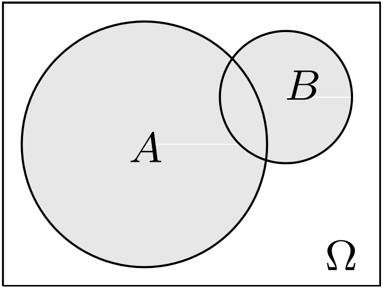
Operations on Sets I

We can visualize operations on sets using so called **Venn diagrams**.



Operations on Sets II

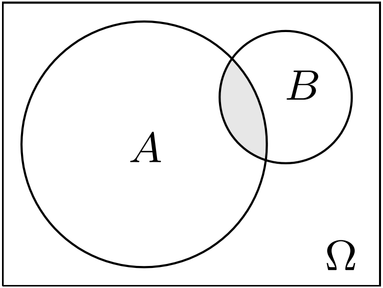
A **union** contains all elements that are either in A or B or in both. Formally, this is $A \cup B = \{x | x \in A \text{ or } x \in B \text{ or both}\}$.



If $A = \{1, 2, 3\}$ and $B = \{3, 4\}$, then $A \cup B = \{1, 2, 3, 4\}$.

Operations on Sets III

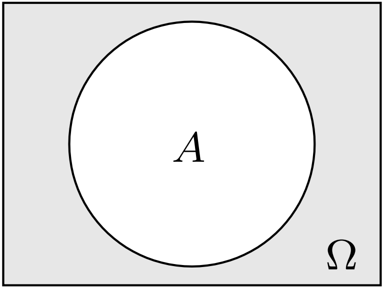
An **intersection** contains all elements that are both in A and B. Formally, this is $A \cap B = \{x | x \in A \text{ and } x \in B\}$.



If $A = \{1, 2, 3\}$ and $B = \{3, 4\}$, then $A \cap B = \{3\}$.

Operations on Sets IV

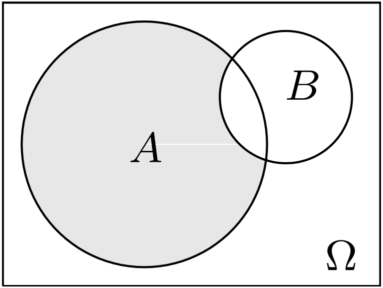
Let there be a **universal set** Ω with the subset A. The **complement** of A is every element of Ω that is not an element of A.
Formally, this is $A^C = \{x|x \notin A \text{ (and } x \in \Omega)\}$.



If $A = \{1, 2, 3\}$ and $\Omega = \{1, 2, 3, 4, 5\}$, then $A^C = \{4, 5\}$.

Operations on Sets V

We can also form **differences** of sets.
$A \setminus B = \{x | x \in A \text{ and } x \notin B\}$.



If $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2\}$, then $A \setminus B = \{3, 4, 5\}$.

Cardinality

The **cardinality** of a set is a measure of the number of elements in the set.
Usually denoted with $|A|$ (alternatives: $n(A)$, $card(A)$ or $\#A$).

If $A = \{1, 2, 3, 4, 5\}$, then $|A| = 5$.

Summary of definitions

$\emptyset$	empty set
$\cup$	union of two sets
$\cap$	intersection of two sets
$\subseteq$	is a subset of
$\subset$	is a strict subset of
$\supseteq$	is a superset of
$\supset$	is a strict superset of

Useful Notation

$\in$	is an element of
$\forall$	for all
$\exists$	there exists
$\Rightarrow$	implies
$\Leftrightarrow$, iff	if and only if
: or $s.t.$	such that
$\equiv$	equivalent to
$\sim$ or $\neg$	not
$\setminus$	without

Laws of Set Theory

Commutative
$A \cup B = B \cup A$ and $A \cap B = B \cap A$

Associative
$(A \cap B) \cap C = A \cap (B \cap C)$ and $(A \cup B) \cup C = A \cup (B \cup C)$

Idempotent
$A \cap A = A$ and $A \cup A = A$

Distributive
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

De Morgan's Laws
$(A \cup B)^C = A^C \cap B^C$ and $(A \cap B)^C = A^C \cup B^C$
$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ and $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

Spaces

Remember: $\mathbb{R}^1$ is the set of real numbers extending from $-\infty$ to ∞, the real number line.

$\mathbb{R}^n$ is an n-dimensional space **("Euclidean space")**, where each of the n axes extends from $-\infty$ to ∞.

Examples:

- $\mathbb{R}^1$ ($\mathbb{R}$) is a line.
- $\mathbb{R}^2$ is a plane.
- $\mathbb{R}^3$ is a 3D-space.

Points in $\mathbb{R}^n$ are ordered n-tuples, where each element of the n-tuple represents the coordinate along that dimension.

Interval Notation for $\mathbb{R}^1$

Open interval: $(a, b) \equiv \{x \in \mathbb{R}^1 : a < x < b\}$

Closed interval: $[a, b] \equiv \{x \in \mathbb{R}^1 : a \leq x \leq b\}$

Half open, half closed interval: $(a, b] \equiv \{x \in \mathbb{R}^1 : a < x \leq b\}$

Neighborhoods: Intervals, Disks, and Balls

We need a formal construct for what it means to be "near" a point **c** in $\mathbb{R}^n$. We call this the **neighborhood** of **c** and represent it by an open interval, disk, or ball, depending on whether n is one, two, or more dimensions, respectively. Given the point **c**, these are defined as

- ϵ-**interval** in $\mathbb{R}^1$: $\{x : |x - c| < \epsilon\}$
  The open interval $(c - \epsilon, c + \epsilon)$.
- ϵ-**disk** in $\mathbb{R}^2$: $\{x : ||x - c|| < \epsilon\}$
  The open interior of the circle centered at **c** with radius ϵ.
- ϵ-**ball** in $\mathbb{R}^n$: $\{x : ||x - c|| < \epsilon\}$
  The open interior of the sphere centered at **c** with radius ϵ.

Interior and Boundary Points

Definition (Interior Point)

The point $\mathbf{x}$ is an interior point of the set S if $\mathbf{x}$ is in S and if there is some ϵ-ball around $\mathbf{x}$ that contains only points in S. The **interior** of S is the collection of all interior points in S.
Example: The interior of the set $\{(x, y) : x^2 + y^2 \leq 4\}$ is $\{(x, y) : x^2 + y^2 < 4\}$.

Definition (Boundary Point)

The point $\mathbf{x}$ is a boundary point of the set S if every ϵ-ball around $\mathbf{x}$ contains both points that are in S and points that are outside S. The **boundary** of S is the collection of all boundary points.
Example: The boundary of the set $\{(x, y) : x^2 + y^2 \leq 4\}$ is $\{(x, y) : x^2 + y^2 = 4\}$.

Open and Closed Sets, Closure

Definition (Open Set)

A set S is called **open** if for each point $\mathbf{x}$ in S, there exists an open ϵ-ball around $\mathbf{x}$ completely contained in S.
Example: $\{(x, y) : x^2 + y^2 < 4\}$

Definition (Closed Set)

A set S is called **closed** if it contains all of its boundary points.
Example: $\{(x, y) : x^2 + y^2 \leq 4\}$

Note: a set may be neither open nor closed.
Example: $\{(x, y) : 2 < x^2 + y^2 \leq 4\}$

Definition (Closure)

The **closure** of set S is the smallest closed set that contains S.
Example: The closure of $\{(x, y) : x^2 + y^2 < 4\}$ is
$\{(x, y) : x^2 + y^2 \leq 4\}$

Bounded Set

Sometimes the definition of a closed set is not sufficient. Consider the following case: the set $(-\infty, 0] \cup [1, \infty)$ is a closed set because its complement $(0, 1)$ is open. However, there is no upper bound to this set.

Definition (Boundedness)

A set $A \subset \mathbb{R}^n$ is **bounded** if it can be contained within an ϵ-ball. That is, there will always be a real-valued number or vector that is outside the set.
Example: any interval that does not have ∞ or $-\infty$ as endpoints; any disk in a plane with finite radius.

Compact Set

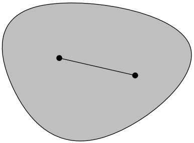
Definition (Compact Set)

A set $A \subset \mathbb{R}^n$ is **compact** if it is closed and bounded.

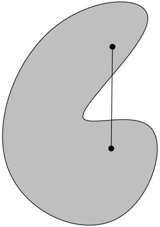
Convexity

Definition (Convex Set)

A set A in $\mathbb{R}^n$ is said to be **convex** iff for each $x, y \in A$, the line segment $\lambda x + (1 - \lambda)y$ for $\lambda \in (0, 1)$ belongs to A. That is, all points on a line connecting two points in the set are in the set.



set is convex



set is not convex

Why Bother with This?

These formal definitions are rather abstract and meaningless at first glance. However, they constitute some very important fundamentals, which ease the life of a scientist. Why is that?

In many applications we can show that some results hold if a set is compact. For example, in game theory we know that (under certain very general assumptions about rationality of persons) amongst a set of possible choices there will always be some alternative which is preferred the most by a person if the set of choices is compact.

In addition, if we know that this set is also convex, we then know that there will be exactly one most preferred alternative.

Beyond this example there are many other applications in political science that use the notion of compact sets.

Set Theory

Functions

What is a function?

Definition (Function)

A **function** or **map**, denoted by $f : X \mapsto Y$, has 3 parts:

- A set X to map from. This set is called the domain of f.
- A set Y to map to. This set is called the co-domain of f.
- A rule for every element $x \in X$, assigning it to some element $y \in Y$. This is written $f(x) = y$

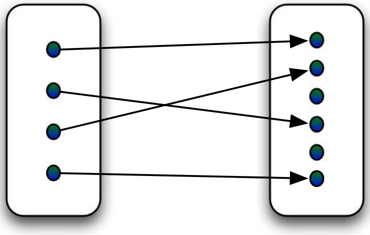
Examples:

- $$\begin{aligned} f &: \{1,2,3\} \to \{3,4,5\} \\ &: x \mapsto x+2 \end{aligned}$$
- $$\begin{aligned} &f : \{1,2\} \to \{1,3\} \\ &f(1) = 1, f(2) = 3 \end{aligned}$$

Linking Sets: Injection, Bijection, and Surjection

Definition (Injection)

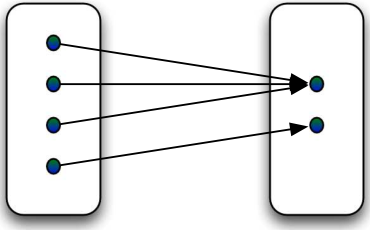
A function f is called **injective** if for every $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Verbally, every element of the codomain Y is linked to at most one element of the domain X.



Linking Sets: Injection, Bijection, and Surjection

Definition (Surjection)

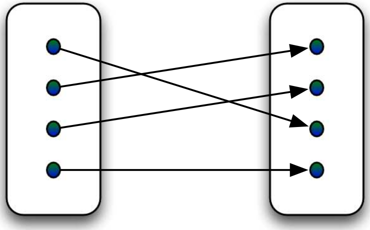
A function f is called **surjective** if for every $y \in Y$ there is an $x \in X$ with $f(x) = y$. Verbally, every element of the codomain Y is linked to at least one element of the domain X.



Linking Sets: Injection, Bijection, and Surjection

Definition (Bijection)

A function f is called **bijective** if it is injective and surjective, i.e. every element of the domain X is linked to one and only one element of the codomain Y and vice versa.



Analysis I

Analysis (Calculus)

Resources:

- Moore/Siegel: Chapters 2, 5-6
- Siegel on Youtube: Lecture 1 Modules 7-9, Lectures 3-4
- Gill: Chapter 5

Rules for Exponentials and Fractions

- $x^0 = 1$
- $x^a = \underbrace{x \cdot x \cdot x \ldots \cdot x}_{a \text{ factors}}$
- $x^a \cdot x^b = x^{a+b}$
- $(x^a)^b = x^{a \cdot b}$
- $(xy)^a = x^a y^a$
- $\frac{1}{x^a} = x^{-a}$
- $\left(\frac{x}{y}\right)^a = \left(\frac{x^a}{y^a}\right) = x^a \cdot y^{-a}$
- $x^{\left(\frac{a}{b}\right)} = (x^a)^{\frac{1}{b}} = \sqrt[b]{x^a}$
- For a^b we say "a raised to the b-th power," "a raised to the power/exponent (of) b," or more briefly "a to the b."
- For $\frac{a}{b}$ we say "a divided by b," "a by b," or "a over b."

Binomial Theorem

- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a-b)^2 = a^2 - 2ab + b^2$
- $(a+b)(a-b) = a^2 - b^2$
- and universally stated: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$; $n \in \mathbb{N}$

Logarithms

- For those with a German background: Please note that in English texts the expression *log* without specification of a base is equal to *ln*, i.e. the natural logarithm!
- $\log_a(1) = 0$
- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$
- $\log_a(x^y) = y \log_a(x)$
- $\log_a(a^x) = x$ and $a^{\log_a(x)} = x$
- Read $\log_a b$ as "the logarithm of b to the base a" or "the base-a logarithm of b"

Quadratic Expressions

Equations of the form $ax^2 + bx + c = 0$ can be solved using the quadratic formula (in German the so-called "Mitternachtsformel")

$$x_{1|2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Equations with one variable

Assume that we want to solve the following equation for x.

$$\begin{aligned}
2\sqrt{x}-3 &= 1 \quad &|+3 \quad & \text{we can add } \ldots \\
2\sqrt{x} &= 4 \quad &|:2 \quad & \ldots\text{divide}\ldots \\
\sqrt{x} &= 2 \quad &|a^2 \quad & \ldots\text{raise to the power...} \\
x &= 4 & & \ldots\text{and much more}
\end{aligned}$$

Equations with several variables

In political science applications solving for one variable oftentimes is not enough. So let us now consider the solution of two simultaneous equations with two variables.

$$\begin{aligned} 2x + 3y &= 4 \quad (1) \\ x - 2y &= 5 \quad (2) \end{aligned}$$

Solve equation (2) for x and insert this into (1):

$$\begin{aligned} x &= 2y + 5 && (2)' \\ 4y + 10 + 3y &= 4 && (2)' \text{ in } (1) \end{aligned}$$

This gives $y = -\frac{6}{7}$. Inserting this into (2)' gives $x = \frac{23}{7}$.

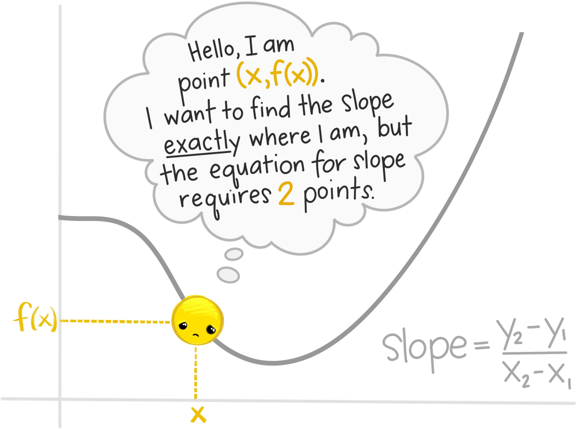
Analysis I

Derivatives

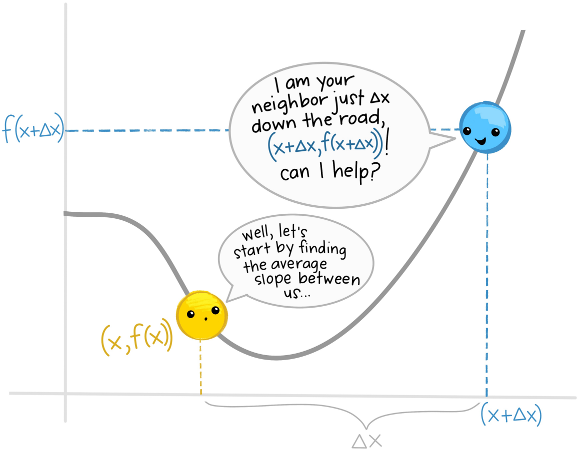
Motivation

- What is the relationship between the level of democracy and economic growth?
- for linear relationships, the information is directly available from the equation - the slope m
- What do we do when we have a non-linear function?
- What is the slope m at some point x_0?

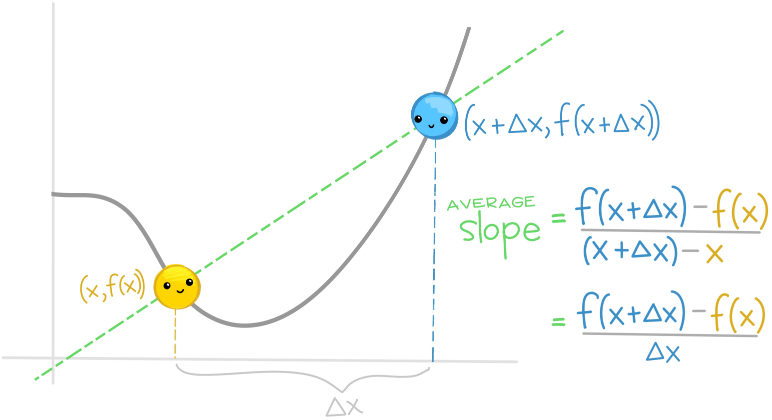
What is a derivative? I



What is a derivative? II



What is a derivative? III

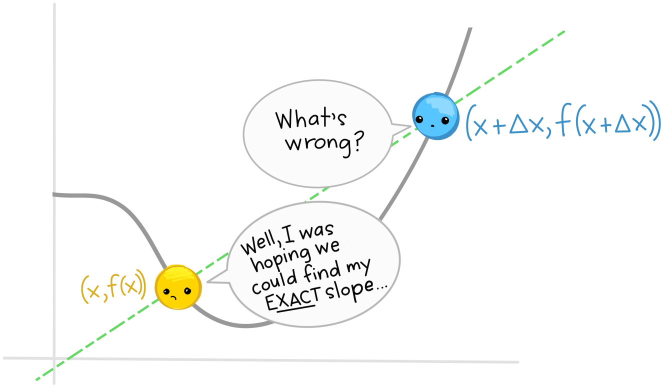


What is a derivative? IV

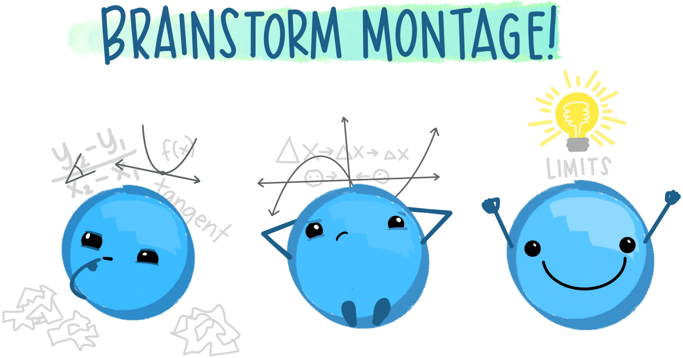
So: the average slope between ANY 2 POINTS on function $f(x)$ separated by Δx is

$$\boxed{m = \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$

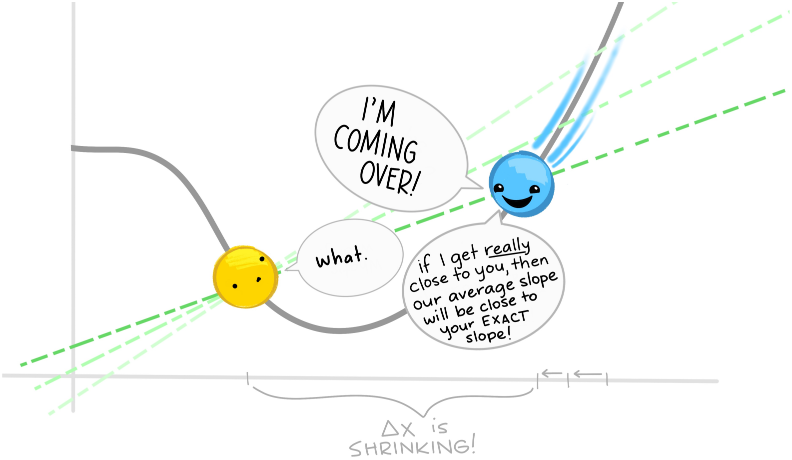
What is a derivative? V



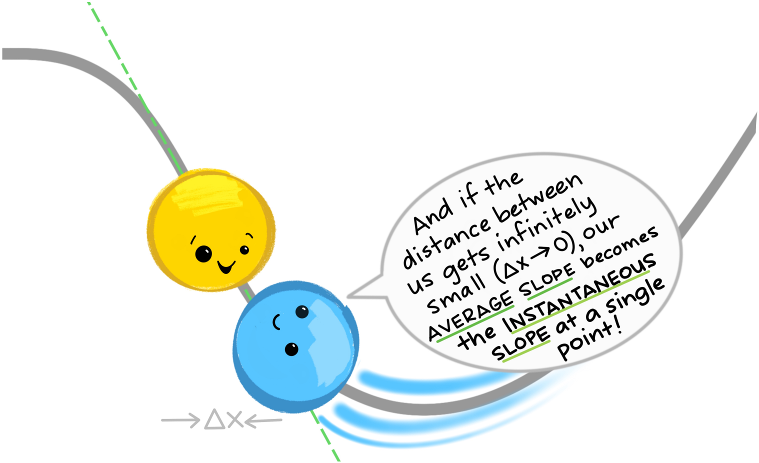
What is a derivative? VI



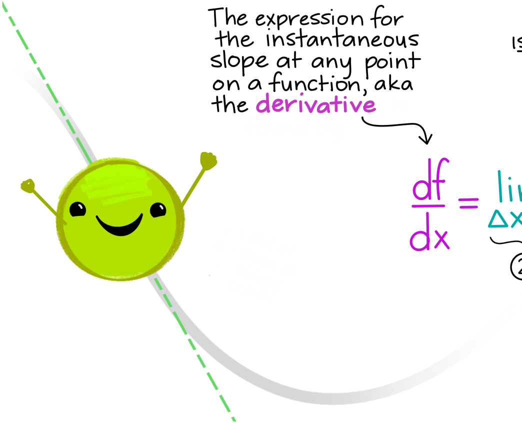
What is a derivative? VII



What is a derivative? VIII



What is a derivative? IX



The expression for the instantaneous slope at any point on a function, aka the derivative

IS FOUND BY:

① Finding an expression for the slope between 2 points separated by Δx...

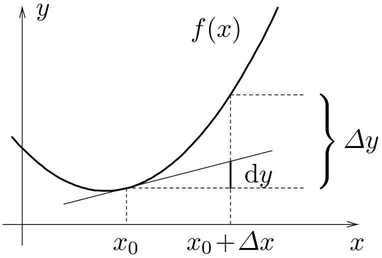
$$\frac{df}{dx} = \underbrace{\lim_{\Delta x \to 0}}_{②} \overbrace{\frac{f(x+\Delta x) - f(x)}{\Delta x}}^{①}$$

② & evaluating that slope as the points get infinitely close together.

What is a derivative? X

We want to estimate the slope of a function at point x_0.

- As a rough estimate we can form the difference quotient $\frac{\Delta y}{\Delta x}$.
- Decreasing Δx continuously brings us closer and closer to the true slope...
- In limit we approach the **derivative** at point x_0.



Illustrations by Allison Horst

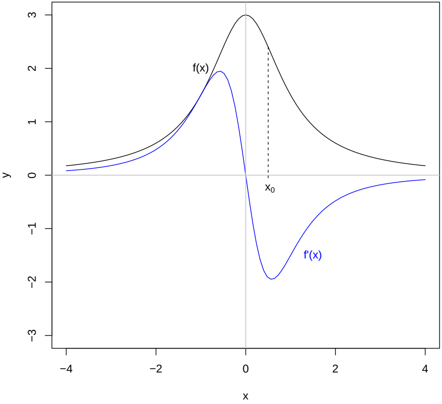
Intuition I

The derivative:

- is a measure of how a function changes as its input changes
- of a function at a chosen input value describes the best linear approximation of the function near that input value
- at a point equals the slope of the tangent line to the graph of the function at that point (linearization of a function for the multivariate case)

Intuition II

- $f(x) = \frac{3}{1+x^2}$
- $f'(x) = -\frac{6x}{(x^2+1)^2}$
- Observations:
  - slope is not a number anymore, but a function (it varies with x)
  - for any x, $f'(x)$ gives us the slope (a value)
  - e.g. $f'(x_0 = 0.5) =$ -1.92



Definition

Definition (Limit of a Function)

Assuming $x, p, c, L \in \mathbb{R}$, the limit of a real valued function f when x approaches p, denoted as $\lim_{x \to p} f(x) = L$, is L if $\forall \epsilon > 0 \exists c > 0, s.t. \forall x, 0 < |x - p| < c \implies |f(x) - L| < \epsilon$.

Note, that if $p = +\infty$ or $p = -\infty$, L is called the asymptote of the function.

Definition

Definition (Derivative)

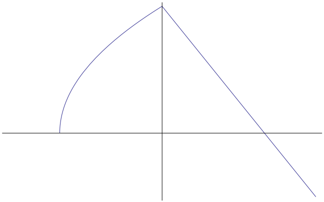
Let $(x_0, f(x_0))$ be a point on the graph of $y = f(x)$. The **derivative** of f at x_0, written $f'(x_0)$, $\frac{df}{dx}(x_0)$, $\frac{dy}{dx}(x_0)$ is the slope of the tangent line to the graph of f at $(x_0, f(x_0))$:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

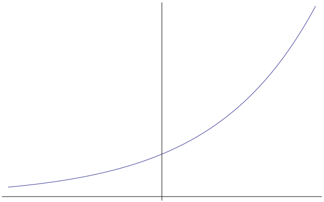
if this limit exists. If this limit exists for every point x in the domain of f, the function is differentiable.

Differentiability

- graph has to be 'smooth' (no gaps, holes, ...)
- if f is differentiable, it must be continuos (converse does not hold)



function is not differentiable

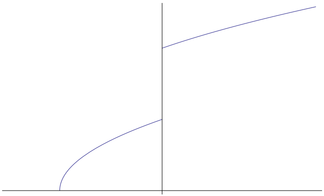


function is differentiable

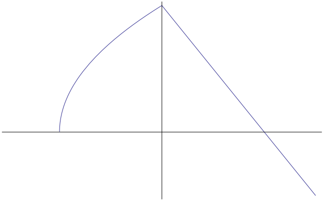
Continuity

Definition (Continuity)

A function f is **continuous** at $x = a$ if $\lim_{x \to a} f(x) = f(a)$

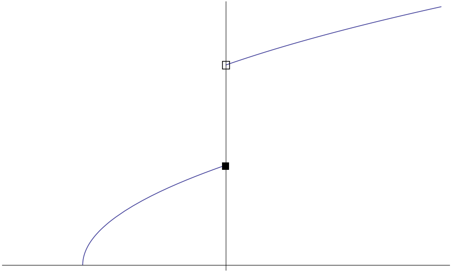


function is discontinuous

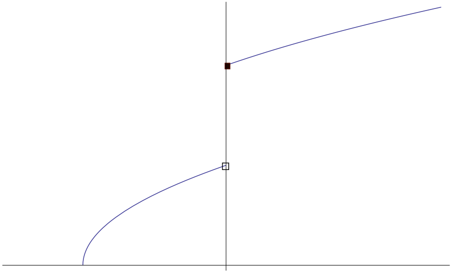


function is continuous

Semi-Continuity



function is lower
(semi-)continuous



function is upper
(semi-)continuous

Analysis I

Rules of Differentiation

Rules of Differentiation I

Rules for Common Functions

- $f(x) = x^a$, then $f'(x) = ax^{a-1}$
- $f(x) = ln(x)$, then $f'(x) = \frac{1}{x}$
- $f(x) = log_a x$, then $f'(x) = \frac{1}{x \ln a}$
- $f(x) = e^{ax}$, then $f'(x) = ae^{ax}$
- $f(x) = a$, where a is a constant, e.g. 1, then $f'(x) = 0$
- $f(x) = a^x$, then $f'(x) = \log_a a^x$
- $f(x) = \frac{1}{x} = x^{-1}$, then $f'(x) = -\frac{1}{x^2}$

Rules of Differentiation II

Sum Rule

- $[f(x) + g(x)]' = f'(x) + g'(x)$
- Example:

$$\begin{aligned} h(x) &= 2x + x^2 \\ h'(x) &= 2 + 2x \end{aligned}$$

Rules of Differentiation II

Product Rule

- $[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- Example:

$$\begin{aligned} h(x) &= 2x \cdot \sqrt{x} \\ h'(x) &= 2 \cdot \sqrt{x} + 2x \cdot \frac{1}{2\sqrt{x}} \end{aligned}$$

Rules of Differentiation III

Quotient Rule

- $\left[\dfrac{f(x)}{g(x)}\right]' = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$
- Example:

$$
\begin{aligned}
h(x) &= \frac{3x}{2 - x^2} \\
h'(x) &= \frac{3 \cdot (2 - x^2) - 3x \cdot (-2x)}{(2 - x^2)^2}
\end{aligned}
$$

Rules of Differentiation III

Chain Rule

- $[f(g(x))]' = f'(g(x)) \cdot g'(x)$
- Example:

$$\begin{aligned} h(x) &= (5x-2)^3 \\ h'(x) &= 3(5x-2)^2 \cdot 5 \end{aligned}$$

Analysis I

Partial Derivatives

Motivation

- What if the relationship between the level of democracy does not only dependent on economic growth, but also on the political institutions?
- We can generalize the concept of a derivative to the multivariate case
- Partial derivates say something about the changes in y given a change in x_i holding all other arguments at some level

Partial Derivatives I

Definition (Partial Derivatives)

Let f be a multivariate function. Then for each variable x_i at each set of points $(x_1^0, ..., x_n^0)$ in the domain of f:

$$\frac{\partial f}{\partial x_i}(x_1^0, ..., x_n^0) = \lim_{h \to 0} \frac{f(x_1^0, ..., x_i^0 + h, ..., x_n^0) - f(x_1^0, ..., x_i^0, ..., x_n^0)}{h}$$

is called the partial derivative, if the limit exists.

Note, that we usually write $\frac{\partial f}{\partial x}$ for partial derivatives and $\frac{df}{dy}$ for derivatives.

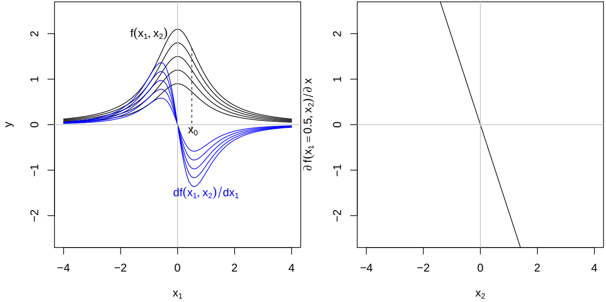
Partial Derivatives II

Example:

$$\begin{aligned} f(x_1, x_2) &= x_1^2 \cdot \ln x_2 \\ \frac{\partial f}{\partial x_1} &= 2x_1 \cdot \ln x_2 \\ \frac{\partial f}{\partial x_2} &= x_1^2 \cdot \frac{1}{x_2} \end{aligned}$$

Intuition

- $f(x_1, x_2) = \frac{3x_2}{1+x_1^2}$
- $\frac{\partial f(x_1,x_2)}{\partial x_1} = \frac{-6x_1x_2}{(x_1^2+1)^2}$
- Observations:
  - slope varies not only with x_1, but also with x_2
  - e.g. $\frac{\partial f(x_1=0.5,x_2)}{\partial x_1} = -1.92x_2$



Second-order Partial Derivatives

Reconsider the example from the last slide

$$
\begin{aligned}
f(x_1, x_2) &= x_1^2 \cdot \ln x_2 \\
\frac{\partial f}{\partial x_1} &= 2x_1 \cdot \ln x_2 \\
\frac{\partial f}{\partial x_2} &= x_1^2 \cdot \frac{1}{x_2} \\
\frac{\partial^2 f}{\partial x_1^2} &= 2 \cdot \ln x_2 \\
\frac{\partial^2 f}{\partial x_2^2} &= -x_1^2 \cdot \frac{1}{x_2^2}
\end{aligned}
$$

Second-order derivatives describe how the slope of the first derivative changes given changes in x.

Mixed Partial Derivatives I

Reconsider the example from the last slide

$$
\begin{aligned}
f(x_1, x_2) &= x_1^2 \cdot \ln x_2 \\
\frac{\partial f}{\partial x_1} &= 2x_1 \cdot \ln x_2 \\
\frac{\partial f}{\partial x_2} &= x_1^2 \cdot \frac{1}{x_2} \\
\frac{\partial^2 f}{\partial x_1 \partial x_2} &= 2x_1 \cdot \frac{1}{x_2}
\end{aligned}
$$

Mixed Partial Derivatives II

Theorem (Young's Theorem)

Suppose that all the m^{th}-order partial derivatives of the function $f(x_1, x_2, \ldots, x_n)$ are continuous. If any of them involve differentiating with respect to each of the variables the same number of times, then they are necessarily equal.

In the case of $f(x_1, x_2)$, that implies for example:

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} \equiv \frac{\partial^2 f}{\partial x_2 \partial x_1}$$

Hessian Matrix I

Because of the importance of the second-order partial derivatives for constrained optimization there does exist a special of collecting them, the so-called **Hessian Matrix**

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial^2 x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial^2 x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial^2 x_n} \end{pmatrix}$$

Application

- Estimation of covariance matrix
- Optimization in maximum likelihood
- ...

Analysis II

Analysis (Calculus)

Resources:

- Moore/Siegel: Chapters 7-8, 15-17
- Siegel on Youtube: Lectures 5-6 and 12-16
- Gill: Chapter 6

Analysis II

Optimization

Motivation for Optimization

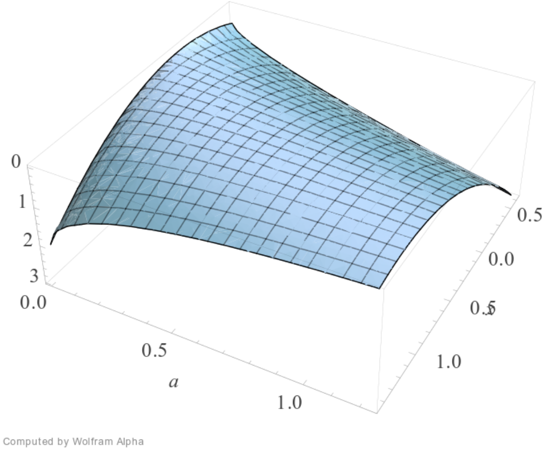
In decision theory we are interested in the decision-making process of an individual.

Let us assume, we have a specified utility function of a person $u(x) = -(x + \sqrt{a})^2$.

We want to know the optimal choice the person can take. How do we do this?

Motivation for Optimization

$$u(x) = -(x + \sqrt{a})^2.$$



Single Variable Optimization - FOC

The first step to get an answer to this problem is to search for the so-called **first-order condition (FOC)**:

$$\frac{df}{dx} \equiv 0$$

- We derive the first line because we know that in an extreme point the slope of the function (i.e. its first derivative) equals zero.
- In our case $\frac{df}{dx} = -2x - 2\sqrt{a}$.
- Solving the equality gives us $x^{\star} = -\sqrt{a}$.
- So now we know that at this point the function either has a (local) maximum/minimum (or a saddle point).

Single Variable Optimization - SOC

Now we need to specify which of the three possibilities applies. We do this by checking the **second-order condition**.

- Local maximum if $\frac{d^2 f}{dx^2}(x^\star) < 0$, i.e. the function is concave
- Local minimum if $\frac{d^2 f}{dx^2}(x^\star) > 0$, i.e. the function is convex
- Saddle point if $\frac{d^2 f}{dx^2}(x^\star) = 0$ and $\frac{d^3 f}{dx^3} \neq 0$.

In our example $\frac{d^2 f}{dx^2}(-\sqrt{a}) = -2$. So we have a local maximum.

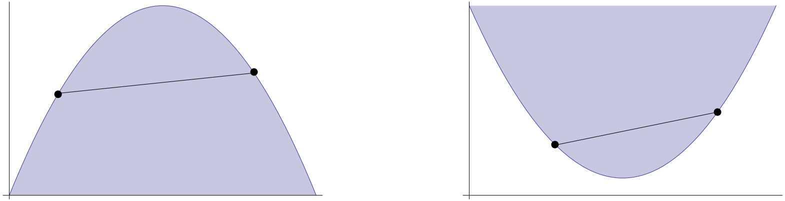
Controlling for the other parts of the function, we find that this is also a global maximum.

Convex, Concave, and Inflection Point

- A function is called **convex** if $\frac{d^2 f}{dx^2} \geq 0$.
- A function is called **concave** if $\frac{d^2 f}{dx^2} \leq 0$.
- A point a as called **inflection point** if $\frac{d^2 f}{dx^2} = 0$ and $\frac{d^2 f}{dx^2}$ changes sign at a.
- If a is an inflection point and $\frac{df}{dx} = 0$, then it is a **saddle point**.

More General Definition of Concavity/Convexity

A function is called concave (convex) if the line segment joining any two points on the graph is below (above) the graph, or on the graph.



concave convex

We can derive the concavity/convexity of functions from the concept of convex sets. A function is called **convex** if the set of all points which are on or above its graph is a convex set. Conversely, a function is called **concave** if the set of all points which are on or below its graph is a convex set.

Bivariate Optimization I

Consider a C^2 function (i.e. a function that is both continuous and twice differentiable) $f(x, y)$ in a convex set S.

Fist-order condition

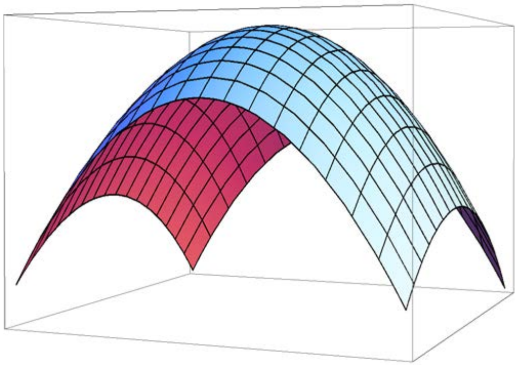
- Find the first-order partial derivatives and equate them to zero.
- Solve the two-equation system for the values of x and y.
- $(x^\star, y^\star)$ is the stationary point.

Second-order condition

- If for all (x, y) in S, $\frac{\partial^2 f}{\partial x^2} \leq 0$, $\frac{\partial^2 f}{\partial y^2} \leq 0$, and $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \geq 0$ then $(x^\star, y^\star)$ is a maximum point for $f(x, y)$ in S.
- If for all (x, y) in S, $\frac{\partial^2 f}{\partial x^2} \geq 0$, $\frac{\partial^2 f}{\partial y^2} \geq 0$, and $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \leq 0$ then $(x^\star, y^\star)$ is a minimum point for $f(x, y)$ in S.

Bivariate Optimization II

Consider the function $f(x, y) = -0.5(x-1)^2 - y^2$.



Bivariate Optimization III

Function $f(x, y) = -0.5(x-1)^2 - y^2$.

The first order condition

$$\begin{aligned} \frac{\partial f}{\partial x} &= -x + 1 \equiv 0 \\ \frac{\partial f}{\partial y} &= -2y \equiv 0 \end{aligned}$$

gives us a stationary point at $x = 1, y = 0$.

Bivariate Optimization IV

The second order condition

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= -1 < 0 \\
\frac{\partial^2 f}{\partial y^2} &= -2 < 0 \\
\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 &= (-1) \cdot (-2) - 0 \geq 0
\end{aligned}$$

tells us that we have a maximum at $x = 1, y = 0$.

Extreme Value Theorem/Weierstrass Theorem

Theorem (Extreme Value Theorem/Weierstrass Theorem)

Suppose the function $f(\mathbf{x})$ is continuous throughout a nonempty, closed and bounded set S in $\mathbb{R}^n$. Then there exists a point $\mathbf{d}$ in S where f has a minimum and a point $\mathbf{c}$ in S where f has a maximum. That is,

$$f(\mathbf{d}) \leq f(\mathbf{x}) \leq f(\mathbf{c}) \text{ for all } \mathbf{x} \in S.$$

You will find the Weierstrass Theorem on page 20 of McCarty and Meirowitz (2007).

Comparative Statics I

Testable predictions of formal models are typically based on **comparative statics**. For example, a researcher might ask...

- ...what happens to the likelihood of the outbreak of civil war if the ethnic diversity of the country increases.
- ...how the level of voter turnout changes as party polarization changes.
- ...how party cohesiveness changes as the level of electoral competitiveness changes?
- ...government public goods provision changes as the size of the winning coalition changes?

More generally: How do changes in the parameters of a model affect the model's solution?

Comparative Statics II

Recall the optimal choice $x^\star = -\sqrt{a}$ of the person with the utility function $u(x) = -(x+a)^2$. How does the optimal choice change as the value of a changes?

$$\frac{dx^\star}{da} = \frac{1}{2\sqrt{a}}$$

An increase of one unit a increases $u(x)$ by $\frac{1}{2\sqrt{a}}$ units, **ceteris paribus**.

Optimization Under Constraints - Problem

So far we have considered decision problems in general. But what about situations in which an agent has to make her decision under given constraints?

Let us consider the following example:
We as a city can decide to allocate our budget between cultural (c) and social (s) affairs. The overall utility function of our city is given by $f(x) = \frac{1}{2}s^2 + (c - \frac{1}{3})^2$. Our budget is constrained as $c + s = 2$.

A method to solve such problems is the so-called **Lagrangian multiplier method**.

Lagrangian Multiplier Method I

In order to solve the maximization problem $\max f(x, y)$ subject to $g(x, y) = c$ we proceed the following way.

1. Write down the Lagrangian
   $\mathcal{L}(x, y) = f(x, y) - \lambda(g(x, y) - c)$, where λ is a constant.
2. Differentiate $\mathcal{L}$ with respect to x and y, and equate the partial derivatives to 0.
3. Solve the system of equations that the two partials form together with the constraint.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} \equiv 0 \\
\frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} \equiv 0 \\
g(x, y) &= c
\end{aligned}$$

Application to our problem

The Lagrangian

$$\mathcal{L}(s, c, \lambda) = \frac{1}{2}s^2 + \left(c - \frac{1}{3}\right)^2 - \lambda(s + c - 2)$$

The system of equations

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial s} &= s - \lambda \equiv 0 \\ \frac{\partial \mathcal{L}}{\partial c} &= 2c - \frac{2}{3} - \lambda \equiv 0 \\ s + c &= 2 \end{aligned}$$

If we solve the system of equations, we get $c = \frac{8}{9}$ and $s = \frac{10}{9}$.

Lagrangian Multiplier Method II

If we compare the Lagrangian method for constrained optimization to the unconstrained optimization, is still something missing?

Yes, theoretically we have to check for the second order-condition.

You find the formulation in Sysdsæter/Hammond (2008) on pp. 506-507.

Advanced Constrained Optimization

There is much more to constrained optimization!

- Multivariable optimization (we need matrix algebra for that!).
- Lagrangian for more than two variables.
- Lagrangian for more than one condition.
- Optimization for inequalities
  $\max f(x, y)$ subject to $g(x, y) \leq c$
  "nonlinear programming" or "Kuhn-Tucker"

See Sysdsæter/Hammond (2008), Chapter 14.

Analysis II

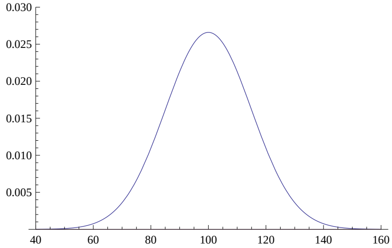
Integration

Motivation I

- probability density functions (p.d.f) are fundamental to statistics
- p.d.f. relate a particular event (x) to a probability (y)
- when we are interested in calculating the probability for a range of events, we need to calculate the area under the curve

Motivation II

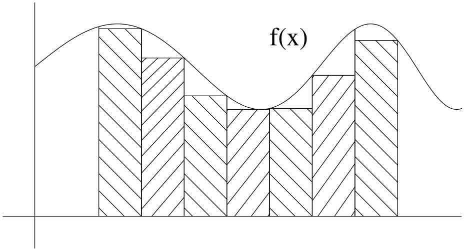
- We know that IQ test scores amongst people of the same age are distributed normally with mean 100 and standard deviation 15.
- What is the probability that a person has a score of more than 120?



It is the area below the normal p.d.f. for $x > 120$ (p$\approx$ 9.12%)

Intuition

- The **indefinite integral** $F(x)$ of a function $f(x)$ is the area between the function and the x-axis.
- We can think of this integral also as the sum of an infinite number of rectangles below the curve!
- Calculating an integral is the reverse process of taking a derivative. For this we sometimes refer to an integral as **antiderivative**.



Definition Integral

Definition (Riemann Integral)

Let f be a continuos function on a closed interval $[a, b]$. Let there be N equal subintervals, each of length $\delta = (b - a)/N$. Let $x_0, x_1, ..., x_N$ be the endpoints of these subintervals, e.i $x_0 = a, x_1 = a + \delta, x_2 = a + 2\delta, ...$ The sum

$$f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) + ... + f(x_N)(x_N - x_{N-1}) = \sum_{i=1}^{N} f(x_i)\delta$$

is the Riemann sum. Taking the limit gives the Riemann integral:

$$\lim_{\delta \to 0} \sum_{i=1}^{N} f(x_i)\delta = \int_a^b f(x)dx$$

Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus (Part I))

Let f be a continuous real-valued function defined on a closed interval $[a, b]$. Let F be the function for all $x \in [a, b]$, by

$$F(x) = \int_a^x f(t)dt$$

Then, F is continuous on $[a, b]$, differentiable on the open interval (a, b), and

$$F'(x) = f(x)$$

for all $x \in (a, b)$.

Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus (Part II))

Let f and F be real-valued functions defined on a closed interval $[a, b]$, such that the derivative of F is f. If f is (riemann) integrable on $[a, b]$ then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Note, that there are infinitely many functions F that have f as their derivative, obtained by adding to F an arbitrary constant. So, we write $\int f(x)dx = F(x) + c$, where c is an arbitrary constant.

Example

$$\begin{aligned}
\int_1^4 x dx &= \Big|_1^4 \frac{1}{2}x^2 \\
&= \frac{1}{2}4^2 - \frac{1}{2}1^2 \\
&= 7.5
\end{aligned}$$

Definite and Indefinite Integral

The difference between an indefinite and a definite integral is the interval of integration.

$$\begin{array}{ll} \int f(x)dx & \text{indefinite integral} \\ \int_a^b f(x)dx & \text{definite integral} \end{array}$$

The numbers a and b are called, respectively, the lower and upper limit of integration.

Properties

Properties (I)

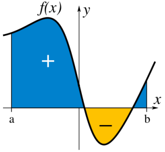
- $\int af(x)dx = a \int f(x)dx$
- $\int [f(x) + g(x)]\, dx = \int f(x)dx + \int g(x)dx$

Properties (II)

Properties (II)

- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^a f(x)dx = 0$
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Caution: Areas between the function and the x-axis which are below the x-axis are subtracted!



Special Cases

Special Integrals

- $\int x^a dx = \frac{1}{a+1} x^{a+1} + c$, where $a \neq -1$
- $\int \frac{1}{x-a} dx = \ln(x-a) + c$, where $x > a$
- $\int e^{ax} dx = \frac{1}{a} e^{ax} + c$, where $a \neq 0$
- $\int a^x dx = \frac{1}{\ln a} a^x + c$, where $a > 0$ and $a \neq 1$

Linear Algebra

Linear Algebra

Resources:

- Moore/Siegel: Chapters 12,13,14.1
- Siegel on Youtube: Lectures 10-11
- Gill: Chapters 3,4

Motivation I

- A statistical model describes how some variables ($x_0 ... x_k$) generate another variable y given some parameters ($\beta_0 ... \beta_k$) and an error term ($\epsilon_1 ... \epsilon_n$), e.g. the linear regression model
- to estimate the parameters, we essentially set up a system of n equations
- each equation describes how each of our n data point was generated, e.g.

$$\begin{aligned} y_1 &= \beta_0 + \beta_1 x_1 + \epsilon_1 \\ y_2 &= \beta_0 + \beta_1 x_2 + \epsilon_2 \\ &\vdots \\ y_n &= \beta_0 + \beta_1 x_n + \epsilon_n \end{aligned}$$

Motivation II

- Linear Algebra gives us the ability (among other things) to answer questions such as:
  - Is there a solution to a system?
  - What is the solution set (e.g. the parameters)?
  - How many solutions are there? What is the space of solutions?
  - Can the system be described by a simpler system of equations?
  - ...
- Matrix notation is a very efficient way to manipulate (simplify) systems of equations

Linear Algebra

Vectors

Vector Spaces and Vectors

Definition (Vector Space)

A vector space V is a nonempty set of objects, called **vectors** denoted with lower case bold letters, on which are defined two operations (addition, multiplication by real scalars), subject to eight axioms:

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ — Commutativity
- $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ — Associativity of vector addition
- $\mathbf{a} + \mathbf{0} = \mathbf{a}$ — Additive identity
- $\mathbf{a} + -\mathbf{a} = \mathbf{0}$ — Existence of an additive inverse
- $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$ — Distributivity of scalar sums
- $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$ — Distributivity of vector sums
- $c(d\mathbf{a}) = (cd)\mathbf{a}$ — Associativity of scalar multiplication
- $1\mathbf{a} = \mathbf{a}$ — Multiplication identity

$\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in V \wedge c, d \in \mathcal{R}$

Euclidean Space

We focus on a special vector space:

- Euclidean space / Cartesian space - $\mathbb{R}^n$
- Euclidean vector: collection of n real numbers either represented as row or column vector:

$$\mathbf{a} = (a_1, a_2, \ldots, a_n) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}'$$

Vector Spaces and Vectors

(continued)

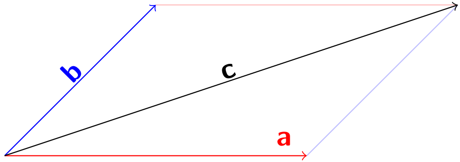
- Terminology: a_i is an **element** or **component**; the vector's **dimension** is the equal to the number of components
- Interpretation of $\mathbf{a}$:
  - line segment connecting the origin $(0, 0)$ with the point $\mathbf{a}$
  - the point $\mathbf{a}$

Vector Operations

Vector addition of vectors with the same dimension is defined as:

$$
\begin{aligned}
(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) &= (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \\
&= \mathbf{a} + \mathbf{b} = \mathbf{c}
\end{aligned}
$$

Graphically ($\mathbb{R}^2$):



Vector Operations

Scalar multiplication of a vector **a** and scalar α is defined as:

$$\alpha(a_1, a_2, \ldots, a_n) = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n)$$

Graphically ($\mathbb{R}^2$):

Vector Norm and Distance

The **norm** (length) of a vector $\mathbf{a} = (a_1, a_2, \dots a_n)$ is defined as:

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 \dots a_n^2} = \sqrt{\sum_{i=1}^{n} a_i^2}.$$

A **normalized vector** has a norm of 1. A **zero vector** has a norm of 0 (note: $\|\mathbf{a}\| = 0 \iff a_i = 0 \forall i$).

Application in $\mathbb{R}^2$: (Euclidean) distance between two points **a**, **b**

$$\|\mathbf{a} - \mathbf{b}\| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} \quad \text{(Theorem of Pythagoras)}$$

Generalized to n-dimensions:

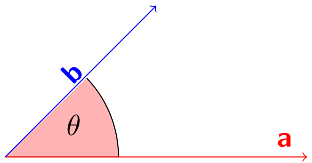
$$\|\mathbf{a} - \mathbf{b}\| = \sqrt{\sum_{i \in n} (a_i - b_i)^2}$$

Dot product

The **inner product** (dot product) of two vectors of equal dimension is defined as:

$$\mathbf{a} \cdot \mathbf{b} = a_1 \cdot b_1 + a_2 \cdot b_2 \ldots a_n \cdot b_n = \sum_{i=1}^{n} a_i b_i$$

Graphically ($\mathbb{R}^2$):



$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos(\theta)$, where θ is the **angle** between the vectors.

Properties

Properties of the Dot Product

If **a**, **b**, and **c** are *n*-vectors and α is a scalar, then

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $(\alpha \mathbf{a}) \cdot \mathbf{b} = \mathbf{a}(\alpha \mathbf{b}) = \alpha(\mathbf{a} \cdot \mathbf{b})$
- $\mathbf{a} \cdot \mathbf{a} > 0 \Longleftrightarrow \mathbf{a} \neq \mathbf{0}$

Linear Algebra

Matrices

Matrix

A **matrix A**, denoted with bold capital letters, is structured into I **rows** and J **columns**. It is said to have the **size** (dimension) $I \times J$. The cells in the matrix are called **elements**.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{21} & a_{22} & \cdots & a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} \end{pmatrix}$$

Matrix Operations

Matrix Addition for two matrices **A** and **B** with the same dimension corresponds to vector addition for each column (or row).

Example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} + \begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 4 \\ 5 & 7 & 9 \\ 8 & 9 & 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} - \begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 2 \\ 3 & 3 & 3 \\ 6 & 7 & 8 \end{pmatrix}$$

Matrix Operations

Scalar Multiplication for a matrix **A** with scalar α corresponds to scalar multiplication of a vector for each column (or row).

Example:

$$2 \times \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{pmatrix}$$

Properties

Properties of Matrices (I)

1. $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$
2. $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$
3. $\mathbf{A}+\mathbf{0}=\mathbf{A}$
4. $\mathbf{A}+(-\mathbf{A})=\mathbf{0}$
5. $(\alpha+\beta)\mathbf{A}=\alpha\mathbf{A}+\beta\mathbf{A}$
6. $\alpha(\mathbf{A}+\mathbf{B})=\alpha\mathbf{A}+\alpha\mathbf{B}$

Matrix Product

Matrix Product of two matrices **A** and **B** with dimension $w \times x$ and $y \times z$ is defined if the number of columns in **A** is equal to the number of rows in **B**, that is, $x = y$. The new matrix has dimension $w \times z$.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1x} \\ a_{21} & a_{22} & \cdots & a_{2x} \\ \vdots & \vdots & \ddots & \vdots \\ a_{w1} & a_{w2} & \cdots & a_{wx} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1z} \\ b_{21} & b_{22} & \cdots & b_{2z} \\ \vdots & \vdots & \ddots & \vdots \\ b_{y1} & b_{y2} & \cdots & b_{yz} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{y} a_{1i} b_{i1} & \sum_{i=1}^{y} a_{1i} b_{i2} & \cdots & \sum_{i=1}^{y} a_{1i} b_{iz} \\ \sum_{i=1}^{y} a_{2i} b_{i1} & \sum_{i=1}^{y} a_{2i} b_{i2} & \cdots & \sum_{i=1}^{y} a_{2i} b_{iz} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{y} a_{wi} b_{i1} & \sum_{i=1}^{y} a_{wi} b_{i2} & \cdots & \sum_{i=1}^{y} a_{wi} b_{iz} \end{pmatrix}$$

Matrix Product

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \times \begin{pmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix} = \begin{pmatrix} 1\cdot 7 + 2\cdot 10 & 1\cdot 8 + 2\cdot 11 & 1\cdot 9 + 2\cdot 12 \\ 3\cdot 7 + 4\cdot 10 & 3\cdot 8 + 4\cdot 11 & 3\cdot 9 + 4\cdot 12 \\ 5\cdot 7 + 6\cdot 10 & 5\cdot 8 + 6\cdot 11 & 5\cdot 9 + 6\cdot 12 \end{pmatrix}$$

$$= \begin{pmatrix} 27 & 30 & 33 \\ 61 & 68 & 75 \\ 95 & 106 & 117 \end{pmatrix}$$

Properties

Properties of Matrices (II)

1. $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
2. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
3. $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$

Note,

- $\mathbf{AB} \neq \mathbf{BA}$
- $\mathbf{A}(\mathbf{B} + \mathbf{C}) \neq (\mathbf{B} + \mathbf{C})\mathbf{A}$

Kronecker Product

If $\mathbf{A}$ is an $w \times x$ matrix and $\mathbf{B}$ is a $y \times z$ matrix, then the Kronecker product $\mathbf{A} \otimes \mathbf{B}$ is the $wy \times xz$ block matrix.

$$
\begin{aligned}
\mathbf{A} \otimes \mathbf{B} &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1x} \\ a_{21} & a_{22} & \cdots & a_{2x} \\ \vdots & \vdots & \ddots & \vdots \\ a_{w1} & a_{w2} & \cdots & a_{wx} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1z} \\ b_{21} & b_{22} & \cdots & b_{2z} \\ \vdots & \vdots & \ddots & \vdots \\ b_{y1} & b_{y2} & \cdots & b_{yz} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1x}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2x}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{w1}\mathbf{B} & a_{w2}\mathbf{B} & \cdots & a_{wx}\mathbf{B} \end{pmatrix}
\end{aligned}
$$

Matrix Transposition

The **Transpose** is defined as a matrix where rows and columns are "interchanged". We denote the transpose of a matrix $\mathbf{A}$ by $\mathbf{A}^T$ or $\mathbf{A}'$.

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

Properties

Properties of Matrices (III)

1. $(\mathbf{A}')' = \mathbf{A}$
2. $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$
3. $(\alpha\mathbf{A})' = \alpha\mathbf{A}'$
4. $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$

Square Matrix

An $i \times j$ matrix $\mathbf{A}$ is called **square matrix** if $i = j$, that is, the numbers of rows and columns are the same.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Symmetric Matrix

A square matrix $\mathbf{A}$ is called **symmetric** if $\mathbf{A} = \mathbf{A}'$. That is, $\mathbf{A}$ is symmetric about its main diagonal. Another way to express this is $a_{ij} = a_{ji} \forall i, j$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 5 \end{pmatrix}' = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

Diagonal Matrix

A square symmetric matrix $\mathbf{A}$ is called **diagonal matrix** if $a_{ij} = 0 \; \forall \; i \neq j$. That is, every element is zero except for the elements on the main diagonal.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Identity Matrix

A square diagonal matrix $\mathbf{A}$ is called **identity matrix I** if the elements on the main diagonal are all equal to one.

$$\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Triangular Matrix

A square matrix $\mathbf{A}$ is called upper (lower) **triangular matrix** if $a_{ij} = 0$ for all $i > j$ ($i < j$), that is, a matrix in which all entries below (above) the main diagonal are 0.

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{pmatrix}$$

Idempotent Matrix

A square matrix $\mathbf{A}$ for which $\mathbf{A} \cdot \mathbf{A} = \mathbf{A}$ is called **idempotent**.

$$\begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix} \times \begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix} = \begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix}$$

The Hessian

Because of the importance of the second-order partial derivatives for constrained optimization there does exist a special way of collecting them, the so-called **Hessian matrix**.

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

Trace

The **trace** of a matrix is the sum of the elements on the main diagonal.

$$\operatorname{tr}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 15$$

Probability Theory

Probability Theory

Resources:

- Moore/Siegel: Chapters 9-11
- Siegel on Youtube: Lectures 7-9
- Gill: Chapter 7

Defintions

- **Experiment**: A probabilistic process that realizes an outcome from a sample space.
- **Sample Space**: S (or Ω), a finite set, the collection of all possible outcomes in an experiment
- **Event**: $A \subseteq S$, a subset from the sample space

Axioms and Definition of Probability

Definition (Probability)

A probability distribution or simply a probability for event A, on a sample space S, is a specification of numbers $Pr(A)$ which satisfy Axioms 1-3 (Kolmogorov probability axioms).

- Axiom 1 (Non-Negativity):

$$Pr(A_i) \geq 0 \ \forall i$$

- Axiom 2 (Normalization):

$$Pr(S) = 1.$$

- Axiom 3 (Additivity):

$$Pr(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} Pr(A_i)$$

with all A_i are disjoint.

Classical Probability of an Event

- Simple Sample Space: $|S| = n$ with $S = \{s_1, ..., s_n\}$
- Event $A \subseteq S$
- Let $|A| = k$

$$Pr(A) = k/n$$

- to determine n and k it is often useful to consider counting rules
- note: classical probability $\neq$ empirical probability $\neq$ subjective probability

Basic Theorems

Let $A, B \subseteq S$:

- $Pr(\emptyset) = 0$
- $Pr(A^c) = 1 - Pr(A)$ where A^c is the complement set to A
- $0 \leq Pr(A) \leq 1$
- $A \subset B \implies Pr(A) \leq Pr(B)$
- $Pr(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} Pr(A_i)$ with all A_i are disjoint
- $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$, where $P(A \cap B)$ is the joint probability of A and B

Note: $Pr(A \cap B)$ is also denoted $Pr(AB)$ or $P(A, B)$

Probability Theory

Combinatorics

Permutation and Combination

	with replacement	without replacement
Permutation (considering sequence)	n^k	$\binom{n}{k}k! = \frac{n!}{(n-k)!}$
Combination (disregarding sequence)	$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Binomial Coefficient

- "n choose k"

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \; \forall \; 0 < k \leq n$$

- Example: How many ways can a voter select three candidates from a field of seven?

$$\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7!}{3! \times 4!} = \frac{7 \times 6 \times 5}{3 \times 2} = 35.$$

Examples I

$k = 2,\ S = \{A, B, C\} \implies n = 3$

	with replacement	without replacement
Permutation (considering sequence)	$\lvert\{AB, BA, BB, AC, CA$ $AA, BC, CB, CC\}\rvert = 9$	$\lvert\{AB, BA, AC, CA$ $BC, CB\}\rvert = 6$
Combination (disregarding sequence)	$\lvert\{AB, AC, BC, AA, BB$ $CC\}\rvert = 6$	$\lvert\{AB, AC$ $BC\}\rvert = 3$

Examples II

Let there be 4 train passengers waiting for tickets. How many sequences are there to sell them their train tickets?
$k = n = 4 \implies \binom{n}{k} k! = 24$

Probability Theory

Conditional Probability

Definition

Definition (Conditional Probability)

Let A,B be two events with probability larger than zero. The conditional probability of A given B is:
$p(A|B) = p(A \cap B)/p(B)$

Interpretation: Given that B occurred, what is the probability for A?

Corollaries

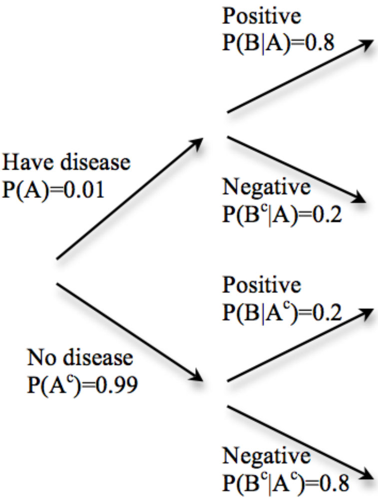
- Multiplication Rule:
  - $p(A \cap B) = p(A|B)p(B)$
- General Product Rule:
  - $P\left(\bigcap_{k=1}^{n} A_k\right) = \prod_{k=1}^{n} P\left(A_k | \bigcap_{j=1}^{k-1} A_j\right)$
- Law of Total Probability
  - Let $A_1, ..., A_k$ be disjoint events and $\bigcup_{i=1}^{k} A_i = S$. For any event B in S and as long as $p(A_j) > 0 \forall j$: $p(B) = \sum_{i=1}^{k} p(A_i)p(B|A_i)$.
- Bayes' Theorem
  - $p(A|B) = \frac{p(B|A)p(A)}{p(B)}$

Example I

Suppose you role a dice, but you can't observe the outcome. What is the probability that you get a 6? Does this probability change when you have been told that the outcome was an even number?

$$\begin{aligned} P(\text{roll a } 6) &= 1/6 \\ P(\text{roll a } 6|\text{even}) &= (1/6)/(1/2) = 1/3 \end{aligned}$$

Example II


Have disease
P(A)=0.01
No disease
P(A^c)=0.99
Positive
P(B|A)=0.8
Negative
P(B^c|A)=0.2
Positive
P(B|A^c)=0.2
Negative
P(B^c|A^c)=0.8

Bayes' Theorem

- first appeared in an essay by Thomas Bayes, 1763
- post-mortem published by Richard Price
- Laplace (1774,1781) provided (independently) most of the relevant analysis
- foundation of Bayesian Statistics, formal modeling of learning, philosophy of scientific progress, ...

Bayes' Theorem

$$\begin{aligned} p(A \cap B) &= p(A|B)p(B) \\ p(A \cap B) &= p(B|A)p(A) \\ p(A|B)p(B) &= p(B|A)p(A) \\ p(A|B) &= \frac{p(B|A)p(A)}{p(B)} \end{aligned}$$

Applied Bayes: Learning Example I

Example: Is a particular coin fair?

- H_1, the event that a head is obtained after tossing
- hypothesis F, the coin is fair; hypothesis $\neg F$, the coin is not fair (has two heads)
- suppose you have no reason to belief more in either of the two hypothesis a-priori
- What is the probability of hypothesis F and $\neg F$ after you tossed the coin and you saw a head?

Applied Bayes: Learning Example II

- **prior** probability about the fairness is $p(F) = p(\neg F) = 0.5$
- if the coin is fair, $p(H_1|F) = 0.5$, if it's unfair $p(H_1|\neg F) = 1$
- the probability for $p(H_1)$ is given by the law of total probability
- **posterior** probability is given by Bayes Theorem:

$$\begin{aligned} p(F|H_1) &= \frac{p(F)p(H_1|F)}{p(F)p(H_1|F)+p(\neg F)p(H_1|\neg F)} \\ &= \frac{(0.5)(0.5)}{(0.5)(0.5)+(0.5)(1)} \\ &= 1/3 \end{aligned} \tag{1}$$

Applied Bayes: Learning Example III

- What is the posterior probability to see head when you toss again (event H_2)?
- now, the prior probability is: $p(F) = 1/3$, $p(\neg F) = 2/3$

$$\begin{aligned} p(F|H_2) &= \frac{p(F)p(H_2|F)}{p(F)p(H_2|F)+p(\neg F)p(H_2|\neg F)} \\ &= \frac{(1/3)(0.5)}{(1/3)(0.5)+(2/3)(1)} \\ &= 1/5 \end{aligned}$$

- for three heads in a row $p(F|H_3) = 1/9$ ...
- this process is called **Bayesian Updating**

Applied Bayes: Statistical Models

- let θ denote a parameter and y the data
- from Bayes' Theorem:

$$\begin{aligned} p(\theta|y) &= \frac{p(y|\theta)p(\theta)}{p(y)} \\ &= \frac{\text{likelihood} \times \text{prior}}{\text{normalizing constant}} \\ &\propto \text{likelihood} \times \text{prior} \end{aligned}$$

- solution to the general problem of inference
- learning about the probability (distribution) of a parameter given the data
- impossible from a frequentist point of view

Probability Theory

Probability Distributions

Random Variable I

Definition (Random Variable)

Let Ω be the sample space for an experiment. A real-valued function that is defined on Ω is called a **random variable**. The set of values the variable might take is the **distribution** of the random variable.

Random Variable II

Definition (Discrete Random Variable)

We say that a random variable X is a **discrete random variable** or that it has a **discrete distribution**, if X can take only a finite number k of different values or, at most, an infinite sequence of different values.

Definition (Continuous Random Variable)

We say that a random variable X is a **continuous random variable** or that it has a **continuous distribution**, if X can take an uncountably infinite number of possible values.

Note, that a random variable is usually denoted with a capital letter, while its realizations are denoted with lowercase letters.

Random Variable - Examples: Coin Toss

- Experiment: toss the coin 10 times.
- Sample space: all possible sequences of of 10 heads and/or tails.
- Random variable: e.g. number of heads, $X =$ Number of Heads

Consider the sequence $q = HHTTTHTTTH$, then $X(q) = 4$. Define another random variable as $Y = 10 - X$, the number of tails. Then, $Y(q) = 6$.

Probability Mass Function

Definition (Probability Mass Function, p.m.f.)

For a discrete random variable X the **probability mass function** of X is defined as a function $f(\cdot)$ such that for every real number x,

$$f(x) = \Pr(X = x) = Pr(s \in \Omega : X(s) = x)$$

Remarks:

- if $x \notin \Omega \implies f(x) = 0$
- if the sequence $x_1, x_2, \ldots$ includes all the possible values of X, then $\sum_{i=1}^{\infty} f(x_i) = 1$.
- $Pr(C \subset \Omega) = \sum_{x_i \in C} f(x_i)$

Discrete Distributions

- Bernoulli: a single coin toss
- Binomial: 'successes' of multiple coin tosses
- Poisson: counts
- ...

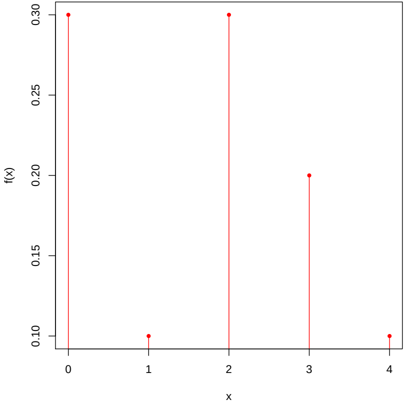
Continuous Distributions

- Normal distribution
- Beta distribution
- Gamma distribution
- χ^2 distribution
- t distribtion
- ...

Example I

A p.m.f. defined as:

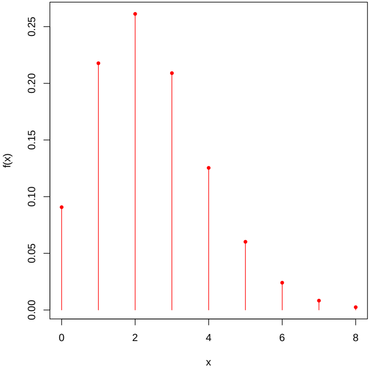
$$f(x) = \begin{cases} 0.3 & \textit{if} \quad x = 0 \\ 0.1 & \textit{if} \quad x = 1 \\ 0.3 & \textit{if} \quad x = 2 \\ 0.2 & \textit{if} \quad x = 3 \\ 0.1 & \textit{if} \quad x = 4 \end{cases}$$



Example II

Let $\lambda \in \mathbb{R}_{>0}$ (intensity), the Poisson p.m.f. is defined as

$$f(x;\lambda) = \begin{cases} \frac{\lambda^x exp(-\lambda)}{x!} & \forall x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$



Comments

- p.m.f. (as in c.d.f. / p.d.f.) have parameters which determine the "shape" of the distribution, e.g. the Poisson p.m.f. has one parameter (λ)
- parameters can be included in the function definition, e.g. $f(x; \lambda)$
- another notation for the Poisson p.m.f. is $X \sim \mathrm{Pois}(\lambda)$ (similar notations exists for common other distributions)
- some authors use $f(X = x)$ instead of $f(x)$ only.

Cumulative Distribution Function

Definition (Cumulative Distribution Function, c.d.f.)

The **cumulative distribution function** $F(\cdot)$ of a discrete or continuous random variable X is the function

$$F(x) = \Pr(X \leq x), \text{ for } -\infty < x < \infty$$

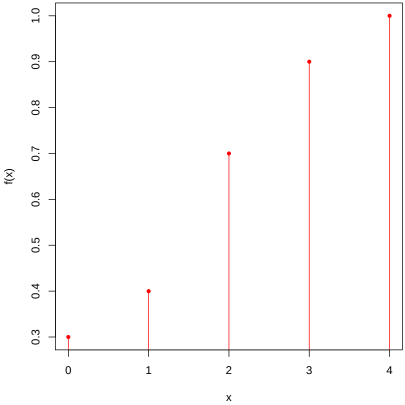
Properties:

- $F(x)$ is nondecreasing as x increases; i.e., if $x_1 < x_2$, then $F(x_1) \leq F(x_2)$.
- $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$.
- c.d.f. is always continuous from the right, i.e. $F(x) = F(x^+)$ at every point x.

Example I

A c.d.f. defined as:

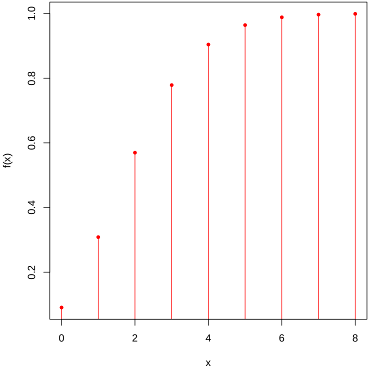
$$F(x) = \begin{cases} 0.3 & if \quad x = 0 \\ 0.4 & if \quad x = 1 \\ 0.7 & if \quad x = 2 \\ 0.9 & if \quad x = 3 \\ 1.0 & if \quad x = 4 \end{cases}$$



Example II

Let $\lambda \in \mathbb{R}_{>0}$ (intensity), the Poisson c.d.f. is defined as

$$F(x) = exp(-\lambda) \sum_{i=0}^{|k|} \frac{\lambda^i}{i!}, \forall k \leq 0$$



Determining Probabilities from the c.d.f.

Let $F(x^-) = \lim_{y \to x} F(y) \forall y < x$ and $F(x^+) = \lim_{y \to x} F(y) \forall y > x$.

For any value:

- x, $\Pr(X > x) = 1 - F(x)$
- x_1 and x_2, such that $x_1 < x_2$, $\Pr(x_1 < X \leq x_2) = F(x_2) - F(x_1)$
- x, $\Pr(X < x) = F(x^-)$
- x, $\Pr(X = x) = F(x) - F(x^-)$

Probability Density Function, p.d.f.

Definition (Probability Density Function)

Let x be a continuous random variable. A p.d.f. is a nonnegative function $f(\cdot)$, defined on the real line, such that:

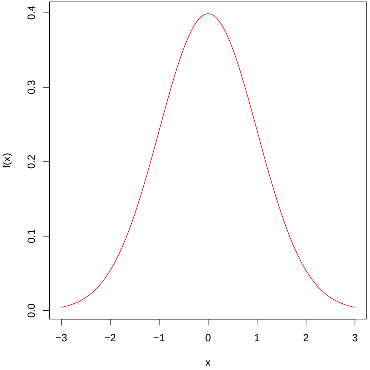
$$f(x) = F(x)'$$

Remarks:

- $f(x) \geq 0, \forall x$
- $\int_a^b f(x)dx = 1$ where a, b are the bounds of the suppport for x

Example I

The p.d.f. of a normal (or Gaussian) distribution is defined as $f(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} exp(-\frac{(x-\mu)^2}{2\sigma^2})$ where $\mu \in \mathbb{R}$ (mean) and $\sigma^2 \in \mathbb{R}_{>0}$ (variance). For the standard normal (picture) $\mu = 0$ and $\sigma^2 = 1$.



Probability Theory

Properties of Distributions

Expectation I

Definition (Expectation)

Let X be a discrete random variable with a p.m.f. $f(\cdot)$. The **expectation** (also: expected value, mean) of X, denoted $E(X)$ is a scalar defined as $E(X) = \sum_x xf(x)$. Similarly, if X is a continuous random variable, the **expectation** is a scalar defined as $E(X) = \int_{-\infty}^{+\infty} xf(x)\,dx$.

Variance

Definition (Variance)

Let X be a random variable with mean $\mu = E(X)$. The variance of X denoted by $Var(x)$ is defined as: $Var(x) = E\big((X-\mu)^2\big)$.

Properties:

- $Var(aX + b) = a^2 Var(X)$
- $Var(X) = E(X^2) - (E(X))^2$
- $Var(X + Y) = Var(X) + Var(Y)$ iff (X, Y) are independent

Remark: For some distributions, the variance is infinite (e.g. Pareto with $\alpha = 0.5$).