University of Mannheim
School of Social Sciences
Mathematics for Political Scientists, Fall 2023
Carlos Gueiros

Exercises: Basics and Set Theory

1. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8\}$ and $C = \{6, 8\}$. Find following:
   (a) $A \cup B$
   (b) $A \cap B$
   (c) $A \cap B^C$
   (d) $B - A$
   (e) $C - B$
   (f) $A \cap C$
2. Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4\}$ and $C = \{a, b, 1, 2\}$. Show that:
   (a) Distributivity: $(A \cap C) \cup (B \cap C) = (A \cup B) \cap C$
   (b) Associativity: $(A \cap B) \cap C = A \cap (B \cap C)$
   (c) De Morgan Laws: $C - (A \cup B) = (C - A) \cap (C - B)$
3. Determine which of the following formulas are true. If any formula is false, find a counterexample to demonstrate this using a Venn diagram.
   (a) $A \setminus B = B \setminus A$
   (b) $A \subseteq B \Longleftrightarrow A \cap B = A$
   (c) $A \cup B = A \cup C \Longrightarrow B = C$
   (d) $A \subseteq B \Longleftrightarrow A \cup B = B$
   (e) $A \cap B = A \cap C \Longrightarrow B = C$
   (f) $A \setminus (B \setminus C) = (A \setminus B) \setminus C$
4. Explain in words why it is true that for any sets A, B, C:
   (a) $(A \cup B) \cup C = A \cup (B \cup C)$
   (b) $(A \cap B) \cap C = A \cap (B \cap C)$
   (c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(d) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

5. Find the interior point(s) and the boundary points(s) of the set $\{x : 1 \leq x \leq 5\}$.

6. Why does every set in $\mathbb{R}$ that is nonempty, closed, and bounded have a greatest member?

7. Which of the following sets are open, closed, or neither?

   (a) $D = \{x \in \mathbb{R}^1 : x = 2 \text{ or } 3 < x < 4\}$
   (b) $A = \{(x, y) \in \mathbb{R}^2 : x^2 \leq y \leq 1\}$
   (c) $B = \{(x, y) \in \mathbb{R}^2 : x^2 < y < 1\}$
   (d) $C = \{(x, y) \in \mathbb{R}^2 : x^2 \leq y < 1\}$
   (e) universal set

8. Sketch the following functions:

   (a) $f(x) = 2$
   (b) $f(x) = 3x - 1$
   (c) $f(x) = x^2 + 2x + 1$
   (d) $f(x) = (x - 3)^{-1}$
   (e) $f(x) = |2x - 2|$
   (f) $f(x) = e^{2x}$
   (g) $f(x) = -\sqrt{x}$

9. Which of the following functions is injective, bijective, or surjective?

   (a) $a(x) = 2x + 1$
   (b) $b(x) = x^2$
   (c) $c(x) = \ln x$
   (d) $d(x) = e^x$

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Exercises: Analysis I

1. Solve the following equations.

   (a) $x^2 - 6x + 8 = 0$

   (b) $(3x-1)^2 - (5x-3)^2 = -(4x-2)^2$

   (c) $\sqrt{x^2 - 9} = 9 - x$

   (d) $\log_x(2x+8) = 2$

   (e) $e^{2x-5} + 1 = 4$

   (f) $\log_2 \frac{2}{x} = 3 + \log_2 x$, where $x > 0$

   (g) $(27)^{2x+1} = \frac{1}{3}$

2. Simplify the following expressions.

   (a) $\frac{4^2 \cdot 6^2}{3^3 \cdot 2^3}$

   (b) $\frac{(x+1)^3(x+1)^{-2}}{(x+1)^2(x+1)^{-3}}$

   (c) $(-3xy^2)^3$

   (d) $\frac{\frac{(x^2)^3}{x^4}}{\left(\frac{x^3}{(x^3)^2}\right)^{-2}}$

   (e) $[(2x+1)(2x-1)]\,(4x^2+1)$

   (f) $\frac{6x^5 + 4x^3 - 1}{2x^2}$

   (g) $\frac{1 + 4x^2 + 6x}{2x - 1}$

   (h) $\frac{x^2 - 5x + 4}{x^2 + 2x - 3} - \frac{x^2 + 2x}{x^2 + 5x + 6}$

3. Show that:

   (a) $\sum_{i=1}^{N}(x_i - \mu_x)^2 = \sum_{i=1}^{N} x_i^2 - N\mu_x^2$. Hint: Note that $\mu_x = \frac{1}{N}\sum_{i=1}^{N} x_i$.

   (b) $\sum_{i=1}^{n}(a_{i+1} - a_i) = a_{n+1} - a_1$.

4. Show that $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.

5. Differentiate the following functions with respect to x.

   (a) $f(x) = 7x^3 - 2x^2 - 5x + 1$

   (b) $f(x) = 0.7x^{-4} + 1.3 - 3.1x^3$

   (c) $f(x) = \dfrac{3x^2 + 1}{2x}$

   (d) $f(x) = \sqrt{4x + 9}$

   (e) $f(x) = \frac{x^{\frac{1}{3}} - 2}{(x^5 - 2)^3}$

   (f) $f(x) = \ln\left(\dfrac{x^2}{x^4 + 1}\right)$

   (g) $f(x) = e^{x^3 + x}$

   (h) $f(x) = \frac{1}{e^x + e^{-x}}$

6. Find the all first and second (mixed) partial derivatives of the following functions.

   (a) $f(x, y) = \ln x \cdot y^2$

   (b) $f(x, y) = \sqrt{2x - y}$

   (c) $f(x, y) = (x + 4y)(e^{-2x} + e^{-3y})$

7. For what value of a is the following function continuous for all x? Is it also differentiable for all x for this value of a?

$$f(x) = \begin{cases} ax - 1 & \text{if } x \leq 1 \\ 3x^2 + 1 & \text{if } x > 1 \end{cases}$$

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Exercises: Analysis II

1. Suppose the function f is defined for all $x \in [-1.5, 2.5]$ by $f(x) = x^5 - 5x^3$.

   (a) Determine for which values of x the value of the function is equal to zero.

   (b) Calculate f'(x) and find the extreme points of f. What is the maximum/the minimum of the function?

   (c) Does the function have inflection points?

2. Which of the following functions of x are convex? Which are concave?

   (a) $f(x) = (2x-1)^6$
   (b) $f(x) = 5x + 7$
   (c) $f(x) = x^5$
   (d) $f(x) = \sqrt{1+x^2}$
   (e) $f(x) = x^5$ for $x \geq 0$
   (f) $f(x) = 5x^2 - x^4$ for $x \geq 1$

3. Appeasement Problem (Ashworth and Bueno de Mesquita, 2006)

   Two states must divide some territory. There is a status quo division, but one state (call is D) is dissatisfied with the status quo. The other state (call is S) is satisfied with the status quo division. S gets one chance to try to appease D by offering it some of the disputed territory. Let x be the fraction of $S's$ territory that it offers. S is uncertain about how dissatisfied D is. S believes that D will accept an offer of x with probability $p(x) = x$. If D accepts the offer, then war is averted and S is left with $1 - x$ of its territory. If D rejects the offer, then there is a war. S believes that it will win a war with probability q. Thus, q can be thought of as $S's$ relative military strength. If S wins the war, then S keeps all of its territory. If S looses the war, it ends up with none of the disputed territory.

   Given all of this, $S's$ maximization problem is given by

   $$\max_x (1-x)x + q(1-x)$$

(a) Find the solution to $S's$ maximization problem.

(b) How does the level of appeasement, x, changes with $S's$ perception of its military strength?

4. Consider the function $f(x) = (x^2 + 2x)e^{-x}$.

   (a) Determine for which values of x the value of the function is equal to zero.

   (b) Calculate f'(x) and find the extreme points of f. What is the maximum/the minimum of the function?

   (c) Does the function have inflection points?

   (d) Sketch the function and specify whether it is convex/concave (in sections).

5. Solve the indefinite integrals:

   (a) $\int \frac{1}{\sqrt{x}} dx$

   (b) $\int e^{-4t} dt$

   (c) $\int x\sqrt{x} dx$

   (d) $\int \frac{1}{x} dx$

   (e) $\int (2x^2 + x - 3) dx$

   (f) $\int \frac{(x^4 + 1)^2}{x^3} dx$

6. Calculate $\int_0^2 (2x^2 + x - 3) dx$. Hint: Make a sketch of the function before.

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Exercises: Linear Algebra

1. Consider the following matrices and vectors.

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 8 & 3 \\ 0 & -1 & 6 \end{pmatrix};\ \mathbf{B} = \begin{pmatrix} -3 & 2 & 4 \\ 2 & 3 & 4 \\ 2 & -4 & 0 \end{pmatrix};\ \mathbf{c} = \begin{pmatrix} 4 & -3 & 2 \end{pmatrix};\ \mathbf{d} = \begin{pmatrix} 3 & 8 \end{pmatrix};$$

$$\mathbf{e} = \begin{pmatrix} 2 & 6 & 9 \end{pmatrix};\ \mathbf{F} = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix};\ \mathbf{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};\ \mathbf{H} = \begin{pmatrix} 5 & 6 & 1 \\ -2 & 7 & 8 \end{pmatrix};$$

$$\mathbf{K} = \begin{pmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{pmatrix}$$

   Do the calculations if possible.

   (a) $\mathbf{M}_1 = \mathbf{A} \cdot \mathbf{B}$
   (b) $\mathbf{M}_2 = \mathbf{A} - \mathbf{B}$
   (c) $\mathbf{M}_3 = \mathbf{B} \cdot \mathbf{F}$
   (d) $\mathbf{M}_4 = \mathbf{A} \cdot \mathbf{c}$
   (e) $\mathbf{M}_5 = \mathbf{c} \cdot \mathbf{A}$
   (f) $\mathbf{m}_6 = \mathbf{d} \cdot \mathbf{c}$
   (g) $\mathbf{m}_7 = 2\mathbf{c} \cdot 3\mathbf{e}$
   (h) $\mathbf{M}_8 = \mathbf{B} \cdot \mathbf{G}$
   (i) $\mathbf{M}_9 = \mathbf{A} \cdot \mathbf{H}$
   (j) $\mathbf{M}_{10} = \mathbf{H}' \cdot \mathbf{F}$

2. What is the size of the following matrices?

   (a) $\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{H}'$
   (b) $\mathbf{c} + \mathbf{e} \cdot \mathbf{H}'$
   (c) $\mathbf{F} \cdot \mathbf{K}$

3. Specify whether the following matrices are square, zero, identity, diagonal or upper/lower triangular matrices and give their dimension as well as their rank.

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 5 & 0 & 8 \\ 0 & 1 & -2 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 7 & 0 \\ 1 & -3 & 9 \end{pmatrix}, \mathbf{E} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 8 & 0 \\ 0 & -5 & 0 \end{pmatrix}$$

4. Is the equation $(\mathbf{F} + \mathbf{G})^2 = \mathbf{F}^2 + 2 \cdot \mathbf{F} \cdot \mathbf{G} + \mathbf{G}^2$ true for any square matrices of the same dimension?

5. Find all 2×2 matrices $\mathbf{A}$ such that $\mathbf{A}^2$ is the matrix obtained from $\mathbf{A}$ by squaring each entry.

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Exercises: Probability Theory

1. Suppose we draw 5 balls from an urn containing 15. How many different sets of drawn balls are there? Consider permutation and combination both with and without repetition.

2. Make a complete list of all the different subsets of the set $\{a, b, c\}$. How many are there if the empty set and the set itself are included? Do the same for the set $\{a, b, c, d\}$. In addition: How many different subsets are there for a set containing n elements?

3. The World Health Organization (WHO) recently detected a new, deadly disease. This so called "CAT-Virus" is currently making its way to Europe and the US. The first symptoms are hallucination and dizziness; quickly followed by pain, fever and shock. Death might follow within 2-4 days. The WHO developed a quick test that everyone can self-administer. The test is 99% accurate. That is, if you have the disease, there is a 99 percent chance that the test will detect it. If you don't have the disease, the test will be 99 percent accurate in saying that you don't. In the general population, 0.1% (that is a tenth of a percentage point) of the people have the disease. When you get a positive test result, what is the probability that you actually have the CAT-Virus?

4. Consider the p.d.f. $f(x) = 2x$ for $0 \leq x \leq 1$.
   (a) Calculate the c.d.f. of $f(x)$.
   (b) Is $f(x)$ a proper p.d.f.?

5. Consider the c.d.f. $G(x) = \frac{1}{9}x^2$ for $0 \leq x \leq 3$.
   (a) Calculate the p.d.f. of $G(x)$, $g(x)$.
   (b) Is $g(x)$ a proper p.d.f.?

6. Consider the p.d.f. $h(x) = \frac{4}{3}(1 - x^3)$ for $0 < x < 1$. Determine
   (a) $\Pr(X < \frac{1}{2})$.
   (b) $\Pr(X > \frac{1}{3})$.

(c) $\Pr(\frac{1}{4} < X < \frac{3}{4})$.

7. Consider the p.d.f. $k(x) = cx^2$ for $1 \leq x \leq 2$. Determine

   (a) Find the value of the constant c.

   (b) Find $\Pr(X > \frac{3}{2})$.